Computational Modular Character Theory

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To Joachim Neubüser
and Herbert Pahlings

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Preface

This book introduces some new ideas for the computer calculation of modular characters of finite groups. These ideas underly the computer algebra system MOC, which has been developed by the authors. Since there have been quite a few results obtained with the aid of MOC, it seems appropriate to publish the concepts and algorithms on which it is built. The sources of the programs of this system are freely available. Should anyone wish to use them for any purpose, apply to any of the authors.

This book consists of six chapters and an appendix. In the first chapter we recall and collect known results about modular character tables of finite groups and introduce the MOC-system. In the second chapter we offer a quick introduction, mostly without proofs, to the modular character theory of finite groups. It is our intention to lead the beginner and non-expert quickly to a state of being able to produce new results with MOC. In Chapter 3 we describe the computational concepts underlying our system. They allow us to transform the problems we are concerned with into problems of solving systems of integral linear equations and inequalities. These concepts are of independent interest as they can also be applied in a generic way to the modular representation theory of finite groups of Lie type. In the fourth chapter we introduce some principal data structures. In particular we suggest a new character table format in which all entries are integers. We also present a previously unpublished result due to H. W. Lenstra on certain integral bases for abelian number fields. The heart of our book is the fifth chapter, where the algorithms are introduced. Some of them should be of independent interest, for example our algorithms for dealing with rational linear equations. Finally, in Chapter 6 we have worked out a non-trivial example in detail, namely the calculation of the 7-modular decomposition numbers of the double covering group of Conway's largest group.

In the appendix we have collected some results obtained in the course of examples throughout the book. These results include the 5-modular decomposition numbers of the Conway group Co_2, including the Brauer character table and the 7-modular decomposition numbers of $2Co_1$. Both results are published for the first time.

Finally we give a bibliography, which is not intended to be exhaustive, of papers and monographs cited in our book and also of some related work.

We hope that even the expert in the field of modular representation theory will find our book interesting, because of our different point of view—we must compute with the objects.

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List of notations

1. Numbers

$\|M\|$	Number of elements of the set M
p	A prime number
$\gcd(n, m)$	Greatest common divisor of the integers n und m
n_p, $n_{p'}$	p-part respectively p-free part of the integer n
$\mathbb{Z}$	Integers
$\mathbb{N}$	Positive integers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{Q}$	Rational numbers
$\mathbb{Q}_n$	$\mathbb{Q}(\zeta_n)$, where ζ_n denotes a primitive n-th roots of unity
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers
$\bar{z}$	Complex conjugate of $z \in \mathbb{C}$
$\mathbb{F}_q$	Finite field with q elements
$\mathcal{G}(L/K)$	Galois group of Galois field extension $K \subseteq L$

2. Matrices

X^t	Transpose of matrix X
$\det(X)$	Determinant of X
E_n	$n \times n$-Identity matrix

3. Groups

G	Finite group
$H \leq G$	H is a subgroup of G
$C_G(X)$	Centralizer of $X \subseteq G$
$N_G(H)$	Normalizer of subgroup H in G
G_p, $G_{p'}$	Set of p- respectively p'-elements of G
$\|x\|$	Order of $x \in G$

6. Rings and modules

R	Commutative ring with 1
R^*	Group of units of R
$\mathrm{rank}_R(X)$	R-rank of the free R-module X
RG	Group algebra of G over R
$\mathrm{Hom}_{RG}(X,Y)$	Set of RG-homomorphisms between the RG-modules X and Y
$[X,Y]$	$\mathrm{rank}_R(\mathrm{Hom}_{RG}(X,Y))$ for RG-modules X and Y
X^G	Induced module
X_H	Restricted module
X^*	RG-module dual to X
(K,R,F)	p-modular system: R: Ring of algebraic integers K: Field of fractions of R F: Residue class field of R of characteristic $p \neq 0$

8. Characters

$\vartheta_{\mathcal{X}}$	Brauer character afforded by the representation $\mathcal{X}$
Φ_φ	Projective indecomposable character corresponding to the irreducible Brauer character φ
PIM	Projective indecomposable character
ϑ^{2+}	Symmetric square of character ϑ
ϑ^{2-}	Skew square of character ϑ
ω_χ	Central character corresponding to $\chi \in \mathrm{Irr}(G)$
$\mathrm{Irr}(G)$	Set of irreducible K-characters of G
$\mathrm{IBr}(G)$	Set of irreducible Brauer characters of G with respect to (K,R,F)
$\mathrm{IPr}(G)$	Set of projective indecomposable characters of G
B	A union of blocks of G
$\mathrm{Irr}(B)$	Set of irreducible K-characters of G belonging to B
$\mathrm{IBr}(B)$	Set of irreducible Brauer characters of G with respect to (K,R,F) belonging to B

$\mathrm{IPr}(B)$	Set of projective indecomposable characters of G belonging to B
$G_0(KG)$	Set of all $\mathbb{Z}$-linear combinations of $\mathrm{Irr}(G)$
$G_0(FG)$	Set of all $\mathbb{Z}$-linear combinations of $\mathrm{IBr}(G)$
$K_0(FG)$	Set of all $\mathbb{Z}$-linear combinations of $\mathrm{IPr}(G)$
$G_0(B)$	Set of all $\mathbb{Z}$-linear combinations of $\mathrm{Irr}(B)$
$G_0(B)^+$	Set of all $\mathbb{N}$-linear combinations of $\mathrm{Irr}(B)$
$G_0(\hat{B})$	Set of all $\mathbb{Z}$-linear combinations of $\mathrm{IBr}(B)$
$G_0(\hat{B})^+$	Set of all $\mathbb{N}$-linear combinations of $\mathrm{IBr}(B)$
$K_0(B)$	Set of all $\mathbb{Z}$-linear combinations of $\mathrm{IPr}(B)$
$\mathcal{C}(G,B)$	Set of class functions of G with values in K, spanned by $\mathrm{Irr}(B)$
$\mathcal{C}_{p'}(G)$	Set of class functions of G which vanish on p-singular elements
$\mathcal{C}_{p'}(G,B)$	Set of class functions of G with values in K, spanned by $\mathrm{IBr}(B)$
ψ_B	Projection of class function ψ to $\mathcal{C}(G,B)$ respectively $\mathcal{C}_{p'}(G,B)$
$\mathcal{C}(G)$	Set of class functions of G with values in K
$[M]$	Matrix of values of $M \subset \mathcal{C}(G)$
χ_H	Class function restricted to subgroup H
χ^G	Class function induced to G
$\bar{\chi}$	Class function complex conjugate to χ
$\bar{M}$	Set of complex conjugates of $M \subset \mathcal{C}(G)$
$\langle\ ,\ \rangle$	Inner product of $\mathcal{C}(G)$
$\langle M,N\rangle$	Matrix of inner products between the elements of the sets $M,N \subset \mathcal{C}(G)$
$\hat{\psi}$	Restriction of class function ψ to the p'-elements
$\widehat{\mathrm{Irr}}(B)$	$\{\hat{\chi} \mid \chi \in \mathrm{Irr}(B)\}$
$\mathbf{D}_B$	Decomposition matrix of B
$d_{\chi\varphi}$	Decomposition number
BS	Basic set of Brauer characters
PS	Basic set of projective characters

$\mathbf{BS}_0$	Special basic set of Brauer characters
$\mathbf{BA}$	Basis of Brauer atoms dual (with respect to $\langle\ ,\ \rangle$) to $\mathbf{PS}$
$\mathbf{PA}$	Basis of projective atoms dual to $\mathbf{BS}$
$\mathbf{PA}_0$	Basis of projective atoms dual to $\mathbf{BS}_0$

Contents

Chapter 1

Introduction

1.1 Modular characters

Representation theory is an effective and indispensable tool in the investigation of finite groups. The table of ordinary irreducible characters of a finite group G encodes an enormous amount of information about G. For example, it can be used in many cases to show that G is a Galois group over some abelian extension of the rational numbers [101]. The beautiful book of Isaacs [70] gives an impression of the innumerable applications of character theory.

Ordinary characters for arbitrary finite groups where introduced by Frobenius in generalization of characters of abelian groups. Frobenius himself already calculated the character tables of the symmetric and alternating groups and of the simple Mathieu groups. To accomplish this work he was led to invent the method of inducing characters [42]. Schur not much later gave another proof for the character tables of the symmetric and alternating groups, and continued his investigations to complete the character tables of their covering groups [112]. The ordinary character tables of the finite simple groups of Lie type are now almost known by the work of Deligne and Lusztig [97]. The famous "Atlas of Finite Groups" [19] collects, among a wealth of other valuable information, the character tables of the most interesting finite simple groups up to a certain order, in particular those for the sporadic simple groups.

Brauer started the investigation of representations of a finite group over a field whose characteristic divides the group order. In [12], he introduced the idea of modular characters, now called Brauer characters, and the idea of decomposition numbers. The knowledge of decomposition numbers is equivalent to the knowledge of the irreducible Brauer characters. One of the main tasks of modular representation theory is to find methods for the calculation of the decomposition numbers for a given finite group.

Let us mention a few of the various applications of modular representation theory. To begin with, it is useful for the determination of the maximal subgroups of a group. For example, building upon Aschbacher's subgroup theorem [2] and some results on modular representations of finite simple groups, Kleidman and Liebeck were able to determine the maximal subgroups of most of the finite classical groups in [87]. There are also applications outside the theory of finite groups, for example in geometry and combinatorics, where finite geometries are studied via their automorphism groups (see for example [109]) or in topology, where topological spaces are investigated via their fundamental groups (see for example [62]). In many potential applications the problem can be reduced to the case of the simple groups, their automorphism groups and covering groups. These groups are known by the classification of the finite simple groups. It would therefore be very desirable to have a description of all irreducible Brauer characters for this class of groups. Such a large collection of examples would doubtlessly lead to new conjectures and theorems in representation theory and then in turn perhaps to new and simpler ways of obtaining these Brauer characters. As Michler has indicated in his survey article [103], such a collection of modular character tables is also extremly useful in checking various famous conjectures of Brauer and Alperin.

1.2 Known results

Let us now describe some of the history of the Brauer character tables which are known up to the present. Brauer and Nesbitt in [12] already calculated the modular irreducible characters for the groups $PSL(2,q)$ in the defining characteristic case, i.e., for fields of characteristic p dividing q. Janko in [81] found the Brauer characters for his first sporadic group J_1 in every odd characteristic and used one of the 11-modular irreducible characters to construct his group as a matrix group of degree 7 over a field of characteristic 11. Fong in [36] determined the 2-decomposition numbers for this group with some ambiguities. In [71] James calculated all but one of the irreducible Brauer characters for the Mathieu groups, the only exception being one character in characteristic 2 for M_{24}. This last character was found only much later with the help of the Meat-Axe [106]. Humphreys in [65] and Benson in [4] investigated the covering groups of the Mathieu groups. The Higman–Sims group has been studied by Humphreys in [66], and Thackray in his thesis [123] determined the 2-modular characters of the McLaughlin group. In [32] Feit introduced some new methods by utilizing Green correspondence in the determination of decomposition numbers, thereby calculating many substantial examples, including some for the triple covering group of the McLaughlin group and the third Janko group J_3. Feit's methods have been extended and led to the results of [56]. By the combined efforts of various authors the decomposition numbers are now completely known for all groups appearing in the Atlas up to page 100, the largest of which is the McLaughlin group. These results are collected in the first part of a modular Atlas of finite groups [82].

1.3 MOC

The computer algebra system MOC (for MOdular Characters) which we are going to describe in this book has been developed by the authors in order to automate some of the more elementary methods known for calculating decomposition numbers. Parts of the material presented here has already been sketched in [99]. An account of the methods we have in mind has been given by James and Kerber in [80, §6.3]. In a sense, modular character theory has been a part of "computational algebra" right from the beginning, and most of the elementary methods of [loc. cit.] have already been described by Brauer and Nesbitt in [12]. Many of the algorithms of MOC which are mostly concerned with systems of integral equations or inequalities are not new, but do not seem to be widely known or used. A subsidiary purpose of our book is therefore to show how powerful these methods can be, particularly the methods of solving systems of integral linear equations by p-adic expansion and the Gomory-algorithm for solving integral linear inequalities.

Some of the concepts developed for MOC have turned out to be of theoretical interest in the investigation of other problems, too. The work of Fong–Srinivasan and Dipper–James has led Geck and one of the authors to the formulation of a theorem on basic sets of groups of Lie type [46]. The concept of basic set as used in this paper was introduced in MOC in order to mechanize the calculation of decomposition numbers for individual finite groups. It is now clear from our experience that many concepts and algorithms can be used also for series of groups of Lie type, since the methods work for generic character tables not just for individual groups.

Some other aspects of our computational work are worth mentioning. First of all, our need for simple data structures for calculation in cyclotomic fields led to a conjecture that abelian number fields always have a nice integral basis consisting of sums (over an orbit) of roots of unity. This conjecture has now been proved by H. W. Lenstra. We are grateful to him for allowing us to present his proof in our book. These special bases for abelian number fields are now also used in the Aachen GAP system [111].

The principal design of MOC is at present different from other computer algebra systems available. MOC consists of a collection of small one-purpose FORTRAN-programs. We have, for example one program for matrix multiplication and one for solving integral linear equations. The various programs communicate via files. More complex tasks are achieved by using Bourne-shell scripts calling a sequence of MOC programs. The shell scripts are an essential part of the MOC system. One advantage of this design is the simple extendability of the system. It allows the easy addition of new commands to MOC. A further advantage is the portability to other UNIX machines. Another important feature of MOC is supported by this design. Every run of a program is recorded in a file which is attached to a particular group and prime. The MOC programs also write their output onto this log-file automatically. This makes it possible to repeat particular calculations and, what is most important, to write down proofs for the results obtained by the machine. Even the proof writing can be done automatically to a certain extent. This feature of MOC has been used to obtain most of the proofs of [56].

We emphasize the fact that MOC only works with Brauer characters and not with representations (c.f. the Meat-Axe [106]) and uses only elementary methods to the extent described in [80, § 6.3]. Nevertheless, in connection with the more advanced methods described in Section 5.5, one can get results which are out of reach of systems working with representations because of the large degrees of the representations involved. On the other hand, for some groups, the Meat-Axe and Condensation (see [110, 98]) are used successfully to solve problems left open by MOC. This applies in particular for calculations in characteristics 2 and 3, where elementary methods are much less efficient than for larger primes. The three systems working together provide a very powerful computational tool for the calculation of irreducible Brauer characters for individual finite groups.

1.4 Series of groups

Apart from individual groups, there is also considerable knowledge on some series of simple groups. For the reason of completeness we briefly sketch the known results. Various authors, among them Robinson, Kerber and James have established a theory of modular representations for the symmetric and alternating groups (see for example [75, 84]). Burkhardt did some work for groups of Lie type; in particular he found all decomposition numbers for the simple groups $PSL(2, q)$ [15], all decomposition numbers for the Suzuki groups in odd characteristics [16], and some decomposition numbers for the Suzuki groups and the unitary groups $U(3, 2^f)$ in even characteristic [16, 17]. Fong in [36] started the investigation for the small Ree groups ${}^2G_2(3^{2m+1})$, but he had to leave some problems open. These were later solved by Landrock and Michler [94]. The work on these groups was completed in [53].

The investigation for the groups of Lie type naturally divides into the case where the modular characteristic is the defining characteristic of the group, i.e., the characteristic of the underlying field, and the case where the two characteristics are different. In the defining characteristic case the representation theory of algebraic groups is relevant, in particular the problem of finding the multiplicities of the simple modules in the Weyl modules. An affirmative answer to Lusztig's conjecture (see [96]) would solve most of the problems for the finite groups of Lie type in this case.

In the non-defining characteristic case, a theory has been evolving starting with a series of papers by Fong and Srinivasan. The highlight of their work is the complete description of all the Brauer trees for the classical groups in [41]. Dipper and James have been developing a representation theory for the general linear groups which has led to the completion of the decomposition numbers for $GL_n(q)$, $n \leq 10$ [79]. Furthermore, some of the decomposition numbers for $G_2(q)$ and $Sp_4(q)$ are known by the work of Hiss, Shamash and White. Geck has determined most of the irreducible Brauer characters for $U(3, q)$ and the Steinberg triality groups ${}^3D(4, q^3)$ in [43, 45]. Finally, the Brauer trees for the Ree groups are known completely [53].

1.5 Special defect groups

In general the problem of finding the Brauer character table of a finite group is considerable harder than that of finding the ordinary irreducible characters. This is partly due to the fact that in contrast to the ordinary character theory there is no inner product on the set of Brauer characters which tells the irreducibility of a Brauer character. However, in particular cases, there are deep theoretical results which can be used to good advantage. If a group is p-soluble, the Fong–Swan theorem states that every irreducible Brauer character is liftable to an ordinary character. This leads immediately to an algorithm for the determination of the irreducible Brauer characters.

Brauer has introduced the concepts of blocks, which leads to a subdivision of the problems. The blocks correspond to the primitive idempotents of the centre of the modular group algebra. The decomposition matrix is a matrix direct sum of the decomposition matrices for the block algebras, and so it suffices to consider each block in turn. Brauer has associated to each block a conjugacy class of p-groups, the defect groups of the block. Their isomorphism type measures the difficulty or complexity of finding the Brauer characters for the block. If the defect groups are trivial, the block is called a block of defect zero, having as decomposition matrix a (1×1)-matrix with single entry 1. If the defect groups are cyclic, the Brauer–Dade theorem imposes strong restrictions on the decomposition numbers. In this case, the decomposition matrix can be encoded in a certain tree, the Brauer tree. There are other defect groups of a particular type, namely dihedral 2-groups and quaternion groups, where all possible decomposition matrices occuring for blocks with these defect groups are known by work of Brauer, Olsson and Erdmann [29].

Broué and Puig have introduced the notion of nilpotent blocks in [13], generalizing the concept of blocks with cyclic defect groups. Puig [108] has given a remarkable extension of the Brauer–Dade theorem. His results, which are neatly explained in [91], imply the knowledge of the decomposition numbers for nilpotent blocks.

It has been conjectured by Donovan that, given a finite p-group D, there are only finitely many matrices which are decomposition matrices for blocks of finite groups with defect groups isomorphic to D. Scopes [113] has shown the conjecture to be true if we restrict the class of groups to symmetric groups. For cyclic defect groups Donovan's conjecture is true by the results of Brauer and Dade. Nevertheless, to write down all decomposition matrices belonging to blocks with a given cyclic defect group, one still uses the classification of finite simple groups (c.f. [31]). It is our opinion that finite simple groups and in particular the sporadic groups will play an important role in the description of decomposition matrices for general defect groups, even if Donovan's conjecture is shown to be true some day. This, we believe, justifies the application of our methods and in particular the usage of MOC.

Chapter 2

Background from modular character theory

In this chapter we introduce our notation and recall some basic facts of character theory. We follow the exposition of Goldschmidt [47] and also Chapter 15 of Isaacs' book [70]. This chapter is intended to give the beginner a quick introduction to those parts of the theory which are essential for the understanding of the following. We give no proofs but references to the appropriate sources. We assume the reader to be familiar with ordinary character theory of finite groups.

Throughout this chapter let G be a finite group, and p a fixed prime number.

2.1 Brauer characters

We use almost the same set-up as Isaacs in Chapter 15 of [70]. Let R denote the ring of algebraic integers over the rationals, and let K be the quotient field of R. Choose a maximal ideal I of R containing p. Finally let $F = R/I$ denote the residue class field of R, a field of characteristic p which is algebraically closed. The canonical epimorphism $R \to F$ induces an isomorphism ρ from the set of $|G|_{p'}$-th roots of unity to F^* (see [70, Lemma (15.1)]), which is used to define Brauer characters as follows.

An element of G of order not divisible by p is called *p-regular* or a

p'-element. An element of order divisible by p is called *p-singular*. The set of all p-regular elements is denoted by $G_{p'}$.

Definition 2.1.1 Let $\mathcal{X} : G \to GL_n(F)$ be an F-representation of G. The *Brauer character* $\vartheta_{\mathcal{X}}$ of $\mathcal{X}$ is a map $\vartheta_{\mathcal{X}} : G_{p'} \to K$. Let $g \in G_{p'}$, and let $\zeta_1, \ldots, \zeta_n$ denote the eigenvalues of $\mathcal{X}(g)$. We then define $\vartheta_{\mathcal{X}}(g) = \sum_{i=1}^{n} \rho^{-1}(\zeta_i)$.

If we apply the canonical epimorphism to $\vartheta_{\mathcal{X}}(g)$, we obtain the trace of the matrix $\mathcal{X}(g)$. We say that the Brauer character $\vartheta_{\mathcal{X}}$ is *afforded* by the representation $\mathcal{X}$. The function complex conjugate to $\vartheta_{\mathcal{X}}$ is also a Brauer character. It is afforded by the representation dual to $\mathcal{X}$. We sometimes think of a Brauer character as being defined on all of G by letting its values be 0 on p-singular elements. This point of view is taken primarily in this and the next chapter in order to simplify notation. In the implementation of the MOC-system a Brauer character is of course only stored as a function on $G_{p'}$.

A Brauer character is called irreducible if it corresponds to an irreducible F-representation of G. It is a basic fact that the set of irreducible Brauer characters of G is finite and linearly independent as functions from G to K ([70, Theorem 15.5]).

Note that there are several maximal ideals I inside R containing p. A different choice of I would in general give another Brauer character for the same F-representation. However, all MOC-calculations are independent of the choice of I. The reader is invited to check this assertion, as we go along.

2.2 The decomposition matrix

The rings (K, R, F) are fixed in the following. They form our *p-modular system*.

The set of ordinary irreducible K-characters of G is denoted by $\mathrm{Irr}(G)$, the set of irreducible Brauer characters by $\mathrm{IBr}(G)$. Since K is a splitting field for the group ring KG, the number of irreducible K-characters equals the number of conjugacy classes of G.

The set of all $\mathbb{Z}$-linear combinations of ordinary characters is a free abelian group with basis $\mathrm{Irr}(G)$. It is isomorphic to the Grothendieck group of the category of finitely generated KG-modules (see [20, Proposition (16.10)]) and is therefore denoted by $G_0(KG)$. The latter fact is not needed in our book; it is only stated to motivate the notation. Similarly, $G_0(FG)$ denotes the set of all $\mathbb{Z}$-linear combinations of irreducible Brauer characters. Elements in these groups are called *generalized* or *virtual characters*. A function on G which is constant on conjugacy classes is called a *class function*. Characters are class functions and thus $G_0(KG)$ and $G_0(FG)$ are subgroups of $\mathcal{C}(G)$, the set of K-valued class functions of G. We let $\mathcal{C}_{p'}(G)$ denote the set of class functions of G with values in K, vanishing on p-singular elements. Then, by our definition, $G_0(FG)$ is contained in $\mathcal{C}_{p'}(G)$.

If $\chi \in G_0(KG)$, we define $\hat{\chi}$ by

$$\hat{\chi}(g) = \begin{cases} \chi(g), & \text{if } g \text{ is } p\text{-regular}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Now comes the important observation. If χ is an ordinary character, $\hat{\chi}$ is a Brauer character. This can be seen as follows. Let $\mathcal{X}$ be a K-representation of G affording the character χ. Let $\tilde{R}$ denote the localization of R at the maximal ideal I. Then $\mathcal{X}$ can be realized over $\tilde{R}$, i.e., there is some representation $\mathcal{Y}$ equivalent to $\mathcal{X}$, such that the matrices representing the group elements have all entries in $\tilde{R}$ ([70, Theorem 15.8]). We remark that $\mathcal{X}$ can even be realized over R ([68, Satz V.12.5]), but this fact is less elementary. The canonical epimorphism $R \to F$ can be extended uniquely to an epimorphism $\tilde{R} \to F$. We thus obtain a representation $\hat{\mathcal{Y}} : G \to GL_n(F)$ by applying this epimorphism to every entry of a representing matrix. Then the Brauer character afforded by $\hat{\mathcal{Y}}$ obviously is $\hat{\chi}$.

We now consider the decomposition homomorphism:

$$\begin{array}{rcl} d : G_0(KG) & \longrightarrow & G_0(FG) \\ \chi & \longmapsto & \hat{\chi} \end{array}$$

The matrix of d with respect to the bases $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$ is called the *decomposition matrix* of G and denoted by $\mathbf{D}$. The entries of $\mathbf{D}$ are the non-negative integers $d_{\chi\varphi}$ defined by

$$\hat{\chi} = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\, \varphi.$$

The purpose of MOC is to support the calculation of decomposition matrices.

It is an important fact, which follows from Brauer's characterization of characters, that the decomposition homomorphism is surjective ([70, Theorem 15.14]). This means that every irreducible Brauer character is a $\mathbb{Z}$-linear combination of $\{\hat{\chi} \mid \chi \in \mathrm{Irr}(G)\}$. The fact that $\mathrm{Irr}(G)$ is a basis of $\mathcal{C}(G)$ now implies that the set of irreducible Brauer characters spans $\mathcal{C}_{p'}(G)$. As $\mathrm{IBr}(G)$ is linearly independent, we obtain that $|\mathrm{IBr}(G)|$ equals the number of p-regular conjugacy classes of G.

2.3 Projective characters and orthogonality relations

We are now going to introduce the projective characters.

Definition 2.3.1 Let $\varphi \in \mathrm{IBr}(G)$. The *projective indecomposable character* corresponding to φ is the ordinary character Φ_φ defined by

$$\Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\, \chi.$$

A projective indecomposable character will be called a PIM in the following. We denote by $\mathrm{IPr}(G)$ the set of all PIMs and by $K_0(FG)$ the group of all $\mathbb{Z}$-linear combinations of $\mathrm{IPr}(G)$. A $\mathbb{Z}$-linear combination with non-negative coefficients of PIMs is called a *projective character*, or a *genuine projective character*.

It is now most important to observe that a PIM vanishes on p-singular classes ([47, (6.9)(a)]). In particular, $K_0(FG) \subseteq \mathcal{C}_{p'}(G)$.

This definition of PIMs is suggested by Brauer's reciprocity law. It can be given without introducing projective modules and some important properties of projective characters can be derived on this more elementary level. Of course, to get deeper results on projective characters one has to give the usual definition via projective modules.

For $z \in \mathbb{C}$ let $\bar{z}$ denote the number complex conjugate to z. The usual inner product on $\mathcal{C}(G)$ is denoted by $\langle\ ,\ \rangle$. It is defined by

$$\langle \lambda, \mu \rangle = \frac{1}{|G|} \sum_{g \in G} \lambda(g)\overline{\mu(g)}. \qquad (2.2)$$

We recall the *orthogonality relations.*

Theorem 2.3.2 (Orthogonality relations for Brauer characters): *The two sets* $\mathrm{IPr}(G)$ *and* $\mathrm{IBr}(G)$ *are dual bases of* $\mathcal{C}_{p'}(G)$ *with respect to* $\langle\ ,\ \rangle$.

Proof. This follows from the orthogonality relations for ordinary characters and the fact that a PIM vanishes on p-singular classes (see [47, (6.10)]). ■

It follows from Theorem 2.3.2 that $\langle \chi, \psi \rangle \in \mathbb{Z}$, if $\chi \in K_0(FG)$ and $\psi \in G_0(FG)$. Furthermore, $\langle \chi, \psi \rangle \in \mathbb{N}$, if χ is a projective character and ψ a Brauer character. The latter observation has the following converse, which will be used in the construction of projective characters.

Proposition 2.3.3 *Let* Ψ *be an ordinary character of* G *which vanishes on* p*-singular classes. Then* $\Psi \in K_0(FG)$. *If, furthermore,*

$$\langle \varphi, \Psi \rangle \geq 0 \qquad \text{for all} \qquad \varphi \in \mathrm{IBr}(G),$$

then Ψ *is a genuine projective character.*

Proof. By the orthogonality relations, $\mathrm{IPr}(G)$ is a K-basis of $\mathcal{C}_{p'}(G)$. Write

$$\Psi = \sum_{\varphi \in \mathrm{IBr}(G)} a_\varphi \, \Phi_\varphi.$$

Then $a_\varphi = \langle \varphi, \Psi \rangle$, which is an integer since φ is a $\mathbb{Z}$-linear combination of restricted ordinary characters. ■

This proposition is used later on to show that certain constructions yield genuine projective characters. The fundamental principle of the MOC-system is to produce projective characters to approximate the columns of the decomposition matrix.

In order to simplify calculations based on the orthogonality relations, we introduce some more notation allowing us to write the orthogonality relations as matrix equations. We first choose representatives $g_1 = 1, g_2, \ldots, g_s$ for the conjugacy classes of G. We assume that the first s' of these elements are p-regular, and that $g_{s'+1}, \ldots, g_s$ are p-singular. Let C denote the $(s \times s)$-diagonal matrix with entries

$|C_G(g_1)|^{-1}, |C_G(g_2)|^{-1}, \ldots, |C_G(g_s)|^{-1}$. If $\mathbf{M} = \{\lambda_1, \ldots, \lambda_m\}$ is a set of class functions of G, we write

$$[\mathbf{M}] = (\lambda_i(g_j))_{1 \leq i \leq m, 1 \leq j \leq s}$$

for the $(m \times s)$-matrix giving the values of the functions in $\mathbf{M}$. For example, $[\mathrm{Irr}(G)]$ is the table of ordinary irreducible characters of G. We always assume that any set of class functions is given in a particular order.

If $\mathbf{N} = \{\mu_1, \ldots, \mu_n\}$ is another set of class functions, then $\langle \mathbf{M}, \mathbf{N} \rangle$ is the matrix $(\langle \lambda_i, \mu_j \rangle)_{1 \leq i \leq m, 1 \leq j \leq n}$ of mutual inner products of the elements of $\mathbf{M}$ with the elements of $\mathbf{N}$. Finally, $\overline{\mathbf{M}} := \{\bar{\lambda}_1, \ldots, \bar{\lambda}_m\}$, where $\bar{\lambda}_i$ is the class function complex conjugate to λ_i.

With these notations we have:

$$\langle \mathbf{M}, \mathbf{N} \rangle = [\mathbf{M}]\, C\, [\overline{\mathbf{N}}]^t. \tag{2.3}$$

Writing E_ℓ for the identity matrix of size ℓ, the two orthogonality relations can be written as follows:

$$[\mathrm{Irr}(G)]\ C\, [\overline{\mathrm{Irr}(G)}]^t = E_s \tag{2.4}$$

$$[\mathrm{IPr}(G)]\ C\, [\overline{\mathrm{IBr}(G)}]^t = E_{s'}. \tag{2.5}$$

2.4 Blocks

Let us recall the fundamental notions of Brauer's block theory. Following Isaacs [70, p. 272], we first introduce the *Brauer graph* of G. Its vertex set is $\mathrm{Irr}(G)$. Two vertices χ and ψ are linked by an edge, if there exists $\varphi \in \mathrm{IBr}(G)$ such that $d_{\chi\varphi} \neq 0 \neq d_{\psi\varphi}$. The set of characters corresponding to a connected component of the Brauer graph is called a *block* of G.

If B is a union of blocks, we write $\mathrm{Irr}(B)$ and $\mathrm{IBr}(B)$ for the sets of irreducible ordinary characters respectively Brauer characters in B. Let $B_1, \ldots, B_b$ denote the blocks of G. Then

$$\mathrm{Irr}(G) = \bigcup_{i=1}^{b} \mathrm{Irr}(B_i),$$

a disjoint union. If $\mathcal{C}(G, B_i)$ denotes the space of class functions spanned by $\mathrm{Irr}(B_i)$, we obtain

$$\mathcal{C}(G) = \bigoplus_{i=1}^{b} \mathcal{C}(G, B_i).$$

For the set of irreducible Brauer characters we have

$$\mathrm{IBr}(G) = \bigcup_{i=1}^{b} \mathrm{IBr}(B_i),$$

again a disjoint union. By our definition of projective characters the ordinary constituents of a PIM are all contained in a unique block. In this sense every PIM belongs to a block. We write $\mathrm{IPr}(B_i)$ for the set of PIMs lying in block B_i. Let $\mathcal{C}_{p'}(G, B_i)$ denote the K-space of class functions spanned by $\mathrm{IBr}(B_i)$. We thus have

$$\mathcal{C}_{p'}(G) = \bigoplus_{i=1}^{b} \mathcal{C}_{p'}(G, B_i).$$

Therefore, the decomposition matrix splits into a direct sum of block decomposition matrices $\mathbf{D}_B$, and we can restrict our attention to the particular blocks, thus greatly simplifying the problem.

The preceding remark of course only makes sense if one can determine the blocks of G from the ordinary character table before knowing the decomposition numbers. This is indeed the case. For $\chi \in \mathrm{Irr}(G)$ let ω_χ denote the central character corresponding to χ defined by

$$\omega_\chi(g) = \frac{|G|\chi(g)}{|C_G(g)|\chi(1)}, \quad g \in G. \tag{2.6}$$

Then $\omega_\chi(g)$ is an algebraic integer and hence is in R for all $g \in G$ (see [70, Theorem 3.7]). Then (see [70, Theorem 15.27]) two characters $\chi, \psi \in \mathrm{Irr}(G)$ are in the same block, if and only if

$$\frac{|G|\chi(g)}{|C_G(g)|\chi(1)} \equiv \frac{|G|\psi(g)}{|C_G(g)|\psi(1)} \pmod{I} \text{ for all } g \in G_{p'}. \tag{2.7}$$

This criterion is independent of the choice of the maximal ideal I of R containing p ([70, Theorem 15.18]).

The easiest blocks to deal with are the so-called blocks of defect 0. They contain exactly one ordinary irreducible character. This is at the same time the unique irreducible Brauer character and the unique PIM in the block. Thus the decomposition matrix of a block of defect 0 is the (1×1)-matrix with single entry 1. An irreducible character χ lies in a block of defect 0, if and only if p does not divide $|G|/\chi(1)$ ([70, Theorem (15.29)]). In particular, if p does not divide the order of G, the decomposition matrix is the identity matrix.

It is often convenient to know the number of irreducible Brauer characters in a block. Since the decomposition map is surjective, the block decomposition matrices have maximal $\mathbb{Z}$-rank and we have:

Remark 2.4.1 Let $\widehat{\mathrm{Irr}}(B) = \{\hat{\chi} \mid \chi \in \mathrm{Irr}(B)\}$. Then the $\mathbb{Z}$-rank of $[\widehat{\mathrm{Irr}}(B)]$ equals $|\mathrm{IBr}(B)|$.

More generally, suppose we have a K-linear map

$$\alpha : \mathcal{C}(G) \longrightarrow \mathcal{C}(G),$$

which leaves $\mathrm{Irr}(B)$ and $\mathrm{IBr}(B)$ invariant and commutes with the decomposition map $d : \mathcal{C}(G) \to \mathcal{C}(G)$. Such homomorphisms arise, for example, from a group automorphism of G or from the duality operation on the modules of G. We are then interested in the number of characters in B fixed by α. We write ${}^{\alpha}\vartheta$ for the α-image of a character ϑ and $\mathrm{Irr}(B)^{\alpha}$, respectively $\mathrm{IBr}(B)^{\alpha}$ for the set of α-invariant elements. The following result generalizes an observation of Brauer and Feit (see [47, (7.14)(b)]), which was communicated to the authors by Knörr.

Proposition 2.4.2 *Let ρ be the character of B defined by:*

$$\rho(g) = \sum_{\chi \in \mathrm{Irr}(B)} \chi(g)\,{}^{\alpha}\chi(g^{-1}), \quad g \in G.$$

Then

$$|\mathrm{Irr}(B)^{\alpha}| = \langle \mathbf{1}_G, \rho \rangle \qquad \text{and} \qquad |\mathrm{IBr}(B)^{\alpha}| = \langle \mathbf{1}_G, \hat{\rho} \rangle.$$

Proof. We have $\langle \mathbf{1}_G, \rho \rangle = \sum_{\chi \in \mathrm{Irr}(B)} \langle \chi, {}^{\alpha}\chi \rangle$, from which the first equation follows. To prove the second, let $\mathbf{D}_B = (d_{\chi\varphi})$ denote the decomposition matrix of B. For $\varphi \in \mathrm{IBr}(G)$ let Φ_φ denote the PIM corresponding to φ. We then have:

$$\langle \mathbf{1}_G, \hat{\rho} \rangle = \sum_{\chi \in \mathrm{Irr}(B)} \langle \chi, {}^{\alpha}\hat{\chi} \rangle =$$

$$\sum_{\chi \in \mathrm{Irr}(B)} \sum_{\varphi \in \mathrm{IBr}(B)} d_{\chi\varphi} \langle \chi, {}^{\alpha}\varphi \rangle = \sum_{\varphi \in \mathrm{IBr}(B)} \langle \Phi_{\varphi}, {}^{\alpha}\varphi \rangle.$$

The result follows. ■

2.5 Generation of characters

We briefly describe the principal methods to produce Brauer characters and projective characters.

2.5.1 Restriction and induction

Suppose H is a subgroup of G. If $\vartheta \in \mathcal{C}(G)$ is a class function of G, the restricted class function ϑ_H is defined by:

$$\vartheta_H(h) = \vartheta(h), \qquad h \in H. \tag{2.8}$$

If ϑ is an ordinary or a Brauer character, then so is ϑ_H. If $\vartheta \in \mathcal{C}(H)$ is a class function of H, the induced class function ϑ^G is defined as follows. For $g \in G$ choose representatives $h_1, \ldots, h_s$ for the H-conjugacy classes contained in the G-conjugacy class of g. Then

$$\vartheta^G(g) = \sum_{i=1}^{s} \frac{|C_G(g)|}{|C_H(h_i)|} \vartheta(h_i). \tag{2.9}$$

The Frobenius reciprocity law asserts that the two operations introduced above are adjoint operations with respect to the inner product $\langle \ , \ \rangle$. This means that, given class functions ϑ, ρ of H respectively G, we have

$$\langle \vartheta^G, \rho \rangle = \langle \vartheta, \rho_H \rangle.$$

Geck has observed that Frobenius reciprocity, together with Proposition 2.3.3, can be used to show that induction takes projective characters to projectives. Let Φ be a projective character of H. Then Φ^H is an ordinary character of G; looking at the definition (2.9), we see that it vanishes on the p-singular conjugacy classes. If φ is an irreducible Brauer character of G, then $\langle \varphi, \Phi^G \rangle = \langle \varphi_H, \Phi \rangle \geq 0$, since φ_H is a genuine Brauer character of H, and Φ is projective. Thus Φ is a genuine projective character of G.

Let ϑ denote a Brauer character of H afforded by some representation $\mathcal{X}$ over F. Then ϑ^G is a Brauer character of G; indeed, it is the Brauer character afforded by the representation of G induced by $\mathcal{X}$ (see [20, §10A, Exercise 10 in §18]). This observation in turn shows that the restriction of a projective character to a subgroup is a projective character.

The following theorem tells us how to find enough projective characters.

Theorem 2.5.1 (Fong [35, Lemma 1]:) *Let* $\mathbf{P}$ *be the set of projective characters of* G *obtained by inducing the PIMs of all maximal subgroups. Then* $K_0(FG)$ *is the* $\mathbb{Z}$*-span of* $\mathbf{P}$.

Proof. This follows from Brauer's induction theorem. See, for example [114, II 4.2,Corollary 2]. ∎

2.5.2 Restriction to p-regular classes

If $\vartheta \in \mathcal{C}(G)$ is a class function of G, let $\hat{\vartheta}$ be defined by (2.1). Since the decomposition map is surjective, the set $\widehat{\operatorname{Irr}}(G) = \{\hat{\chi} \mid \chi \in \operatorname{Irr}(G)\}$ spans $G_0(FG)$ over the integers.

2.5.3 Restriction to blocks

Let B denote a union of p-blocks of G. Let $\psi = \sum_{\chi \in \operatorname{Irr}(G)} m_{\psi,\chi}\chi \in G_0(KG)$. Then the restriction of ψ to B is the generalized character

$$\psi_B = \sum_{\chi \in \operatorname{Irr}(B)} m_{\psi,\chi}\chi. \tag{2.10}$$

Thus ψ_B is just the projection of ψ from $\mathcal{C}(G)$ onto $\mathcal{C}(G, B)$, the space of class functions spanned by $\operatorname{Irr}(B)$.

Similarly, if ϑ is a generalized Brauer character, ϑ_B is the projection of ϑ onto $\mathcal{C}_{p'}(G,B)$. Thus if $\vartheta = \sum_{\varphi\in\mathrm{Irr}(G)} d_{\vartheta,\varphi}\varphi$, then

$$\vartheta_B = \sum_{\varphi\in\mathrm{Irr}(B)} d_{\vartheta,\varphi}\varphi. \tag{2.11}$$

We remark, that in each case we have to know a basis of the spaces $\mathcal{C}(G,B)$ respectively $\mathcal{C}_{p'}(G,B)$ in order to calculate the projections, but of course any basis will do.

2.5.4 Tensor products

The product of any two characters is again a character. This is true for ordinary and Brauer characters. For projective characters the following more general statement is true. Namely, if ϑ is a Brauer character and Ψ is a projective character, then the tensor product $\vartheta\Psi$ is a projective character. As Geck has observed, a proof of this fact can be given similar to those in § 2.5.1 by just using Proposition 2.3.3.

This statement for projective characters is extremely important for the purposes of MOC. It is used mainly in the case when Ψ is an ordinary irreducible character of defect 0. These are of course the projective characters which are readily available. The above statement means in particular, that the product of two ordinary irreducible characters is projective, if one of the factors is a defect 0 character. The projectives obtained in this way only depend on the character table of G and not on the set of subgroups of G. If we are able to determine the irreducible Brauer characters of G by using only projectives of this sort, the decomposition numbers are determined by the character table.

2.5.5 Symmetrizations

This paragraph is somewhat less elementary than the preceding ones, since the facts we are going to describe do not seem to follow by just considering characters.

Let χ be an ordinary or modular character of G. The symmetric and skew square of χ are the class functions

$$\chi^{2+}(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2))$$

respectively

$$\chi^{2-}(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2)), \qquad g \in G.$$

The symmetric or skew square of a character is again a character, afforded by the symmetric respectively skew power of the module affording χ (if $p \neq 2$). This observation can be generalized to yield, for every positive integer r and every partition λ of r a *symmetrized character* χ^λ.

We now outline this method following James [76]. Let V denote the SG-module affording the character χ. Here, S is one of the fields K or F. Let r denote a positive integer satisfying $r \leq n$, where $n = \mathrm{Dim}_S(V)$. Let $V^{\otimes r}$ denote the r-fold tensor product of V on which G acts diagonally. Now V and hence $V^{\otimes r}$ is a module for the general linear group $GL(V)$ in its natural action on V and thus every $GL(V)$-section (a quotient module of a submodule) of $V^{\otimes r}$ gives rise to an SG-module by way of restriction. If the characteristic p of S is 0 or larger than r, then $V^{\otimes r}$ is a semisimple $GL(V)$-module. For every partition λ of r there is an irreducible $GL(V)$-module V_λ, the *Weyl module* corresponding to λ, and $V^{\otimes r}$ is a direct sum of the Weyl modules. If p is positive and less than or equal to r, then $V^{\otimes r}$ still has a filtration by Weyl modules V_λ, but they are no longer direct summands. Furthermore, in general the Weyl modules are not irreducible anymore. Every Weyl module V_λ has a unique top composition factor F_λ. The other composition factors of V_λ are of the form F_μ for partitions μ which are strictly smaller than λ in the dominance order of partitions. Thus, if the partitions of r are arranged in reverse lexicographic order, the matrix $M_{r,p}$ giving the multiplicities of the F_λ in the V_μ is lower unitriangular.

Let ψ^λ denote the character of G afforded by the Weyl module V_λ. It can be calculated with the help of the character table of the symmetric group S_r. Let $\rho = (\rho_1, \rho_2, \ldots, \rho_r)$, $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_r \geq 0$, be a partition of r. Let $C(\rho)$ denote the centralizer in S_r of an element with cycle type ρ, and let $[\lambda](\rho)$ denote the value of the irreducible character of S_r corresponding to λ on an element with cycle type ρ. Then, for p-regular $g \in G$,

$$\psi^\lambda(g) = \sum_{\rho \vdash r} \frac{1}{|C(\rho)|}[\lambda](\rho) \prod_{i=1}^{r} \psi(g^{\rho_i}). \tag{2.12}$$

If p is non-zero and does not exceed r, than we can improve on the above symmetrization formula, since the Weyl modules are not irreducible in general. Let $\Sigma r, p = M_{r,p}^{-1}[\mathrm{Irr}(S_r)]$. This matrix is called by James an "extended p-modular character table of S_r". The character φ^λ of G afforded by the simple module F_λ can then be calculated by the formula (2.12) with $[\lambda]$ replaced by the corresponding row of $\Sigma r, p$.

In [76] James has calculated the matrices $M_{r,p}$ and $\Sigma_{r,p}$ for $r \leq 6$ and $p = 2, 3$. For example, labelling the rows and columns of the following matrices with the partitions of 3 in reverse lexicographic order, we have

$$M_{3,3} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and

$$\Sigma_{3,3} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix}.$$

If ψ is a 3-modular character of a group G of degree at least 3, the three 3-modular symmetrizations are the following:

$$\varphi^{(1,1,1)}(g) = \frac{1}{6}\psi(g)^3 - \frac{1}{2}\psi(g^2)\psi(g) + \frac{1}{3}\psi(g^3)$$

$$\varphi^{(2,1)}(g) = \frac{1}{6}\psi(g)^3 + \frac{1}{2}\psi(g^2)\psi(g) - \frac{2}{3}\psi(g^3)$$

$$\varphi^{(3)}(g) = \psi(g^3).$$

The formulae of James have been implemented in the CAS system [104]. In his thesis [49], Grabmeier has extended the calculations of James to the case of $r \leq 9$ and $p = 2, 3$.

Chapter 3

Computational Concepts

We are now going to introduce the notions of basic sets and atoms and derive some of their most important properties. These notions are fundamental for the algorithms of MOC, in particular those which try to prove the irreducibility of a Brauer character or the indecomposability of a projective character. They allow us to translate some of the problems occuring during the calculation of decomposition matrices into the language of Linear Algebra. Furthermore, the MOC methods generalize all elementary methods for calculating decomposition numbers described in the literature (see [80, Section VI.3] for a collection af these methods). They can be systematically applied and are mechanized so as to be suitable for the treatment with computers.

We also formulate two fundamental computational problems which we believe to be of independent interest.

It turned out that these concepts, originally designed for application to sporadic groups, could also be applied successfully to the study of modular characters of finite groups of Lie type (see [52]). They allow to deal with all groups of a fixed rank and Dynkin type simultaneously. The reason for this is that in a sufficient general situation we can find basic sets which can be described independently of the underlying field (see [46]).

3.1 Basic sets

We continue with the notation introduced in the preceding section. Here, we let B denote a fixed union of p-blocks of G. Let $\mathrm{Irr}(B) = \{\chi_1, \ldots, \chi_n\}$. The subgroups of $G_0(KG)$, $G_0(FG)$ and $K_0(FG)$ generated by the characters in B are denoted by $G_0(B)$, $G_0(\hat{B})$ and $K_0(B)$ respectively. We write $G_0(B)^+$ for the set of genuine characters of $G_0(B)$, and use an analogous notation for Brauer and projective characters.

Definition 3.1.1 A *basic set of Brauer characters* **BS** (*basic set of projective characters* **PS**) is a $\mathbb{Z}$-basis of $G_0(\hat{B})$ (of $K_0(B)$) consisting of Brauer (projective) characters.

We emphasize the fact that a basic set consists of proper characters, in contrast to the usage in the literature. One of the reasons is the experience that one should never give away the information of a character being proper. We invest a great deal of time to reach the aim of having basic sets at every stage of the calculations. In particular, we introduce an algorithm which for a given set of proper characters determines a $\mathbb{Z}$-basis of their span consisting again of proper characters.

The first question we investigate, is the following. Given a set of Brauer characters, how can we decide whether it is a basic set?

Lemma 3.1.2 *Let $S \subset G_0(\hat{B})^+$. Then S is a basic set if and only if S is linearly independent over $\mathbb{Z}$, and every ordinary character of B, restricted to the p-regular classes, is a $\mathbb{Z}$-linear combination of elements of S.*

Proof. The necessity of the condition is clear. The converse follows from the surjectivity of the decomposition homomorphism. ∎

Example 3.1.3 Here, and in the following examples, we denote irreducible characters by their degrees, which in our examples identifies them uniquely. Consider the principal 7-block B of Conway's simple group Co_1. It contains the following 27 ordinary irreducible characters:

$$\left\{\begin{array}{rrrrr}
\underline{1}, & \underline{276}, & \underline{299}, & \underline{17250}, & \underline{80730}, \\
\underline{94875}, & \underline{822250}, & \underline{871884}, & \underline{1821600}, & \underline{2055625}, \\
\underline{9221850}, & \underline{16347825}, & \underline{21528000}, & \underline{21579129}, & \underline{24667500}, \\
31574400, & \underline{57544344}, & \underline{66602250}, & \underline{85250880}, & \underline{150732800}, \\
\underline{163478250}, & 191102976, & \underline{207491625}, & 215547904, & 219648000, \\
299710125, & 326956500 & & &
\end{array}\right\}.$$

If we restrict these to the 7-regular classes, the subset **BS** of the 21 underlined characters is linearly independent, and the six remaining characters not in **BS** can be written as $\mathbb{Z}$-linear combinations of these according to the following table (we omit the hat on top of the degrees):

31574400	.	.	.	.	.	−1	.	.	1	.	.	−1	1	.	1	.	.	.	.	.	.
191102976	1	.	−1	.	−1	.	1	.	−1	.	−1	1	.	1	.	.	.	.	.	1	.
215547904	.	.	.	−1	−1	1	1	1	−1	.	−1	1	−1	.	.	.	1	.	.	1	.
219648000	1	.	.	1	.	.	.	.	.	.	.	1	.	1	−1	.	−1	−1	1	.	1
299710125	.	.	.	1	.	−1	−1	.	1	−1	1	.	.	.	.	.	−1	.	1	.	1
326956500	.	−1	1	−1	.	1	.	1	−1	1	−1	1	−1	1	−1	1	.	−1	.	1	1

Here, the columns correspond to the 21 characters in increasing order of their degrees. Thus the block contains exactly 21 irreducible Brauer characters and our 21 characters are a basic set.

We have seen how to prove for a given set of Brauer characters that it is a basic set. Now the question arises, of how to do the same for projective characters. By the theorem of Fong cited in § 2.5.1 we know that in principle enough projective characters can be obtained to span the space $K_0(B)$.

We assume that we have constructed a set of projective characters and selected from these a maximal linearly independent subset **PS**. How can we decide whether or not **PS** is a basic set?

Lemma 3.1.4 *Let* **BS** *and* **PS** *be two sets of Brauer characters respectively projective characters. We assume that each contains exactly* $|\mathrm{IBr}(B)|$ *elements. Let*

$$U = \langle \mathbf{BS}, \mathbf{PS} \rangle \tag{3.1}$$

be the matrix of their mutual scalar products. Then **BS** *and* **PS** *are basic sets if and only if* U *is invertible over* $\mathbb{Z}$.

Proof. We have

$$[\mathbf{BS}] = U_1\,[\mathrm{IBr}(B)] \tag{3.2}$$

and

$$[\mathbf{PS}] = U_2^t\,[\mathrm{IPr}(B)]. \tag{3.3}$$

with integral matrices U_1 and U_2. From (2.3) and the orthogonality relations we obtain:

$$U = [\mathbf{BS}]\,C\,[\overline{\mathbf{PS}}]^t = U_1[\mathrm{IBr}(B)]\,C\,[\overline{\mathrm{IPr}(B)}]^t U_2 = U_1 U_2.$$

Now **BS** respectively **PS** are bases if and only if U_1 and U_2 are invertible over $\mathbb{Z}$ and the assertion follows. ∎

Example 3.1.5 Consider again the principal 7-block of Co_1. Let **BS** be as in Example 3.1.3. After some calculations, which we do not want to comment on right here, (but see Chapter 6), we have found a set **PS** of projective characters, such that $U = \langle \mathbf{BS}, \mathbf{PS} \rangle$ is the matrix given below, where the columns correspond to the projective characters. Now $\det U = 1$, so that **PS** is indeed a basic set of projectives by Lemma 3.1.4.

Ψ :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
276	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
299	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
17250	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
80730	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
94875	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
822250	1	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
871884	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
1821600	1	1	.	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
2055625	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
9221850	1	.	.	.	.	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.
16347825	.	.	.	1	2	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.
21528000	1	.	.	.	.	1	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.
21579129	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.
24667500	1	.	.	.	1	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.
57544344	2	.	1	1	.	.	1	.	.	.	1	.	.	1	.	1	.	.	.	.	.
66602250	.	.	.	2	1	.	.	.	1	1	.	.	.	1	.	.	2	1	1	.	.
85250880	2	.	1	2	.	.	.	.	.	1	.	.	.	2	.	1	1	.	.	.	.
150732800	.	.	.	1	.	.	.	.	.	1	.	.	1	1	.	1	1	1	.	1	1
163478250	3	1	.	1	.	1	.	.	.	.	1	.	1	1	.	1	1	.	.	1	1
207491625	1	.	.	1	.	.	.	.	.	1	.	.	1	3	.	.	3	1	1	1	.

3.2 Atoms

Definition 3.2.1 A $\mathbb{Z}$-basis **BA** (**PA**) of $G_0(\hat{B})$ ($K_0(B)$) is called a *system of atoms of Brauer (projective) characters*, if every Brauer (projective) character is a linear combination with non-negative coefficients of elements of **BA** (**PA**). A *Brauer atom (projective atom)* is a generalized Brauer (projective) character, which is a member of a system of atoms.

In contrast to the definition of basic sets, we do not insist that our system of atoms consists of proper characters. The next result shows how to obtain a system of projective atoms from a basic set of Brauer characters. Of course, Brauer characters and projective characters can be interchanged in the formulation of this and many of the following results.

Lemma 3.2.2 *Let* **BS** *be a basic set of Brauer characters. If* **PA** *denotes the basis of* $K_0(B)$ *dual to* **BS** *with respect to the bilinear form* $\langle\ ,\ \rangle$, *then* **PA** *is a system of projective atoms.*

Proof. We have

$$[\mathbf{BS}] = U\,[\mathrm{IBr}(B)],$$

where the entries of U are non-negative integers, and $\det U = \pm 1$. From

$$[\mathbf{PA}]\,C\,[\overline{\mathbf{BS}}]^t = E_m$$

and the orthogonality relations we obtain

$$[\mathrm{IPr}(B)] = U^t\,[\mathbf{PA}],$$

and thus our assertion. ∎

Given **BS**, it is not necessary for our purposes, though in principle possible, to know the elements of the dual basis **PA** explicitly as class functions. Suppose we are given two basic sets **BS** and **PS**. The corresponding dual bases of atoms are denoted by **PA** and **BA**. Suppose furthermore, that **P** and **B** are two sets of projective characters respectively Brauer characters. Then to express the elements of **P** in terms of the basis **PA** is equivalent to calculating the inner products of the elements of **P** with the characters in **BS**. Namely, let $V = \langle \mathbf{P}, \mathbf{BS} \rangle$. Then

$$[\mathbf{P}] = V\,[\mathbf{PA}].$$

Similarly, if $U = \langle \mathbf{PS}, \mathbf{B} \rangle$, then

$$[\mathbf{B}] = U^t\,[\mathbf{BA}]. \tag{3.4}$$

In particular, if we take **BS** for the set **B** of Brauer characters in (3.4), we obtain the elements of **BA** as class functions by inverting the matrix U^t.

Before we continue with our exposition, we give an easy but nevertheless very useful application of the notions introduced so far.

Lemma 3.2.3 *Suppose* **PA** *is a system of projective atoms. If* Φ *is a projective character contained in* **PA**, *then* Φ *is indecomposable.*

Proof. Let $\mathbf{PA} = \{\Phi_1, \ldots, \Phi_m\}$ with $\Phi = \Phi_1$. Furthermore, let $\Psi_1, \ldots, \Psi_m$ denote the PIMs. We write

$$\Psi_i = \sum_{j=1}^{m} z_{ij} \Phi_j.$$

Then all $z_{ij} \geq 0$, and their matrix is invertible. Since Φ is projective, we can write

$$\Phi = \sum_{i=1}^{m} x_i \Psi_i$$

with $x_i \geq 0$. Therefore

$$\Phi_1 = \Phi = \sum_{j=1}^{m} \left(\sum_{i=1}^{m} x_i z_{ij} \right) \Phi_j.$$

From this it follows that $x_i = 0$ for all $i \neq i_0$ and $x_{i_0} = 1$, i.e. $\Phi = \Psi_{i_0}$. ■

Example 3.2.4 The projective character Ψ_{15} of Example 3.1.5 is an atom, since it has inner product 1 with 21579129, but 0 with all the other characters of **BS**. Hence it is a PIM.

Of course there is an analogous statement for Brauer characters. The above observation allows to prove indecomposability of projective characters and irreducibility of Brauer characters. It is, however, only a very special case of a much more general procedure, which will be described in Chapter 5.

3.3 The first fundamental problem

We are now ready to formulate the first fundamental problem of our theory. We assume that we are given two basic sets **BS** and **PS**. The corresponding dual bases of atoms are denoted by **PA** and **BA**. We are interested in all matrices which can possibly be decomposition matrices with respect to the given information. First we calculate the matrix U of the scalar products of the characters of **BS** with the characters of **PS**. Writing

$$[\mathbf{BS}] = U_1\,[\mathrm{IBr}(B)] \qquad \text{and} \qquad [\mathbf{PS}] = U_2^t\,[\mathrm{IPr}(B)], \tag{3.5}$$

we have, as in the proof of Lemma 3.1:

$$U = U_1\,U_2. \tag{3.6}$$

Notice that by our definition of basic sets, U_1 and U_2 have non-negative entries. We write $X \geq 0$ to indicate that every entry of the matrix X over the integers is non-negative. Having **BS** at our hands, we know how to write the restriction of the ordinary characters to the p-regular classes in terms of the characters in **BS**, say

$$[\widehat{\mathrm{Irr}}(G)] = V\,[\mathbf{BS}].$$

Knowing U_1 is then equivalent to knowing the decomposition matrix:

$$\mathbf{D}_B = V\,U_1.$$

Thus the decomposition matrix is obtained from a solution of

$$U = U_1\,U_2, \qquad U_1, U_2 \text{ unimodular }, \qquad U_1, U_2 \geq 0. \tag{3.7}$$

Two solutions (U_1, U_2) and (U_1', U_2') of (3.7) are called equivalent, if there is a permutation matrix X such that $U_1' = U_1 X$ and $U_2' = X^t U_2$. Equivalent solutions lead to the same set of Brauer characters arranged in a different order. If (3.7) has more than one equivalence class of solutions, then we cannot decide which of them determines the decomposition matrix without further information.

In many cases, however, we have additional conditions which drastically reduce the number of solutions of (3.7). Namely, if we are given a set of Brauer characters $\mathbf{B}$ and a set of projective characters $\mathbf{P}$, we can write:

$$[\mathbf{B}] = V\,[\mathbf{BS}] = (V\,U_1)\,[\mathrm{IBr}(B)]$$

and

$$[\mathbf{P}] = W\,[\mathbf{PS}] = (W\,U_2^t)\,[\mathrm{IPr}(B)]$$

with integer matrices V and W. The solutions U_1 and U_2 of (3.7) must therefore satisfy the following conditions:

$$V\,U_1 \geq 0, \quad \text{and} \quad W\,U_2^t \geq 0. \tag{3.8}$$

So we have the first fundamental problem:

Fundamental Problem I: Given $V \in \mathbb{Z}^{s\times m}$ and $W \in \mathbb{Z}^{t\times m}$, and a unimodular matrix $U \in \mathbb{N}^{m\times m}$, find all solutions $U_1, U_2 \in \mathbb{Z}^{m\times m}$ of

$$U = U_1\,U_2, \qquad U_1, U_2 \geq 0.$$

satisfying

$$V\,U_1 \geq 0, \quad \text{and} \quad W\,U_2^t \geq 0.$$

Every solution of the fundamental problem leads to a possible decomposition matrix by equation (3.5): $[\mathrm{IBr}(B)] = U_1^{-1}\,[\mathbf{BS}]$. This shows that at the present state of knowledge the fact that the determinant of the Cartan matrix $\mathbf{D}_B^t\,\mathbf{D}_B$ is a power of p does not impose any restrictions on the solutions of the fundamental problem (cf. the remarks in [80, p. 273]).

We do not intend to solve this fundamental problem in its full generality. Usually the number of solutions is much too large to be of any practical use. Instead we gradually improve on the two basic sets using the conditions (3.7) and (3.8) leading to a matrix U with smaller entries. How this is done in practise with the help of methods of integral linear programming will be described in Chapter 5.

Example 3.3.1 We continue with our example of the principal 7-block of Co_1. Let $\mathbf{B}$ be the set of the six ordinary characters which are not contained in $\mathbf{BS}$, restricted to the 7-regular classes of Co_1. Then, of course, the matrix V is just the one given in Example 3.1.3.

We also have some projective characters which cannot be expressed in terms of **PS** of Example (3.1.5) entirely with non-negative coefficients. The corresponding matrix W is as follows.

$$W = \begin{bmatrix}
1 & -1 & -1 & 1 & 1 & \cdot & \cdot & 3 & -2 & 2 & -1 & 3 & 1 & 1 & \cdot & 3 & 4 & 7 & -9 & 2 & -8 \\
1 & -1 & 1 & \cdot & \cdot & 2 & 6 & 5 & 1 & \cdot & 2 & \cdot & 5 & \cdot & \cdot & 8 & -1 & \cdot & 2 & -4 & 4 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & -1 & \cdot & \cdot & 1 & 3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 & 1 & -1 & -1 & \cdot \\
1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & -1 & \cdot & \cdot & 1 & 1 & 1 & -1 & 1 & -2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & -1 & 1 & 2 & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & 2 & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & -1 & -2 & 1 & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 2 & -2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 2 & 1 & -1 & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & 2 & \cdot & \cdot & 1 & \cdot & -1 & 1 & -2 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 & 1 & \cdot & -2 & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & 2 & \cdot & \cdot & \cdot & 1 & -2 & \cdot & -2 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 & \cdot
\end{bmatrix}$$

With these data, there are only 4 solutions of the fundamental problem I, namely, up to a permutation of the columns, U_1 is one of the matrices given in Table 3.1. There, $a, b \in \{0, 1\}$. We do not intend to give a proof of this fact here. An indication of how this is done is given in Chapter 5.

If we are left with a small number of non-equivalent solutions of the fundamental problem, we can distinguish cases. We take each solution in turn and pretend that it leads to the true decomposition matrix. Then we tensor all irreducible Brauer characters with themselves, and write the resulting characters in terms of the irreducibles. If this leads to expressions with negative coefficients, our assumption was wrong and we consider the next solution.

In principle one could extend the fundamental problem by adding the conditions arising from tensor products. Suppose $\mathbf{BS} = \{\vartheta_1, \ldots, \vartheta_m\}$ and $\mathbf{PS} = \{\Psi_1, \ldots, \Psi_m\}$. Let $a_{ijl} = \langle \vartheta_i \vartheta_j, \Psi_l \rangle$. For $i = 1, 2$, let $U_i^{-1} = (\underline{u}^i_{rs})_{r,s}$. Then the solution (U_1, U_2) which determines the decomposition

Table 3.1: Possible decomposition matrices for the principal block of Co_1

1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
276	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
299	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
17250	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
80730	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
94875	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
822250	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
871884	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
1821600	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
2055625	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
9221850	.	.	.	.	.	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.
16347825	.	.	.	.	1	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.
21528000	.	.	.	.	.	1	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.
21579129	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.
24667500	1	.	.	.	1	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.
57544344	.	.	1	.	.	.	1	.	.	.	1	.	.	.	.	1	.	.	.	.	.
66602250	.	.	.	1	.	.	.	.	1	a	.	.	.	.	.	.	1	.	.	.	.
85250880	.	.	1	1	.	.	.	.	.	1	.	.	.	.	.	1	.	1	.	.	.
150732800	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	1	.	.
163478250	.	1	.	.	.	1	.	.	.	.	1	.	1	b	.	.	.	.	.	1	.
207491625	.	.	.	1	.	.	.	.	.	a	.	.	.	.	.	.	1	1	.	.	1

matrix must satisfy:

$$\sum_{r=1}^{m}\sum_{s=1}^{m}\sum_{t=1}^{m} a_{rst}\underline{u}^1_{ir}\underline{u}^1_{js}\underline{u}^2_{tl} \geq 0, \qquad \text{for all } i,j,l,$$

since the expression on the left hand side equals $\langle \varphi_i\varphi_j, \Phi_l\rangle$, where $[\varphi_i]$ respectively $[\Phi_i]$ is the i-th row of U_1^{-1} $[\mathbf{BS}]$ respectively $U_2^{-t}[\mathbf{PS}]$.

In the ideal case, the matrix U in (3.1) is the identity matrix. Then we have solved our problem.

Proposition 3.3.2 *Let* **BS** *and* **PS** *be two basic sets such that the matrix U of their mutual scalar products is the identity matrix. Then* **BS** *consists of the irreducible Brauer characters and* **PS** *of the PIMs.*

Proof. This immediately follows from (3.6), (3.2) and (3.3). ■

As already mentioned above, MOC tries to reach at this final stage by improving the two basic sets **BS** and **PS** step by step by making use of the conditions (3.8).

We finally indicate a proof of a fact mentioned in [80, Remark 6.3.34], namely that the information gained by inducing Brauer characters from a subgroup is the same as that obtained by restricting projective characters to this same subgroup. It is not really necessary to use the language of basic sets to do so, but it can be used to quantify the information.

Let H be a subgroup of G. We assume that the decomposition matrix $\mathbf{D}_H$ of H is known. Let F denote the G-H-induction-restriction matrix, i.e.,

$$[\mathrm{Irr}(G)_H] = F\,[\mathrm{Irr}(H)],$$

where $\mathrm{Irr}(G)_H$ denotes the set of restrictions to H of the irreducible characters of G. Then, if $\mathrm{Irr}(H)^G$ is the set of characters of G induced from the irreducible characters of H, we have

$$[\mathrm{Irr}(H)^G] = F^t\,[\mathrm{Irr}(G)],$$

by Frobenius reciprocity. Let **BS** be a basic set for the Brauer characters of G. Since the decomposition map is surjective, there is an integral matrix Y such that

$$[\mathbf{BS}] = Y\,[\widehat{\mathrm{Irr}}(G)].$$

We choose such a Y and fix it in the following. Then

$$[\mathrm{IBr}(G)] = X\,[\widehat{\mathrm{Irr}}(G)],$$

with $X = U_1^{-1}Y$. By the surjectivity of the decomposition homomorphism and the definition of the PIMs, this implies that

$$[\mathrm{Irr}(G)] = X^t\,[\mathrm{IPr}(G)].$$

Now $[\mathrm{IPr}(H)] = \mathbf{D}_H^t\,[\mathrm{Irr}(H)]$, and so

$$[\mathrm{IPr}(H)^G] \quad = \quad \mathbf{D}_H^t\,[\mathrm{Irr}(H)^G]$$

$$\begin{aligned} &= \mathbf{D}_H^t\, F^t\, X^t\, [\mathrm{IPr}(G)] \\ &= \mathbf{D}_H^t\, F^t\, X^t\, U_2^{-t}\, [\mathbf{PS}] \\ &= \mathbf{D}_H^t\, F^t\, Y^t\, U^{-t}\, [\mathbf{PS}]. \end{aligned}$$

Thus the information gained by inducing the projective characters from H is the following condition for U_2 coming from (3.8)

$$(\mathbf{D}_H^t\, F^t\, Y^t\, U^{-t})\, U_2^t \geq 0.$$

On the other hand, restricting the irreducible Brauer characters of G down to H we obtain

$$\begin{aligned} [\mathrm{IBr}(G)_H] &= X\, [\widehat{\mathrm{Irr}}(G)_H] \\ &= X\, F\, [\mathrm{Irr}(H)] \\ &= U_1^{-1}\, Y\, F\, \mathbf{D}_H\, [\mathrm{IBr}(H)] \\ &= U_2\, U^{-1}\, Y\, F\, \mathbf{D}_H\, [\mathrm{IBr}(H)], \end{aligned}$$

giving exactly the same conditions as above.

3.4 The second fundamental problem

In general, the two matrices of (3.8) have many rows. But if one of these rows is a sum of others, then the row obviously contains no condition at all and can be deleted. As our experience shows, a quick procedure to throw away these obsolete rows is of enormous help. For example, in the course of the calculation of the 7-modular characters of the Conway group $2.Co_1$ we had to produce more than 10000 projective characters in order to find the few which are of any use. This leads us to the formulation of our second fundamental computational problem. Before we do this, we give an ad hoc definition, which is only needed to formulate the problem.

Definition 3.4.1 Let W be a finite subset of $\mathbb{Z}^{1\times s}$, i.e., a set of rows of integers of length s. A subset $W_0 \subseteq W$ is called *essential*, if for all $w \in W$ there are non-negative integers n_v, $v \in W_0$, such that $w = \sum_{v\in W_0} n_v v$.

If the zero vector is not in the positive $\mathbb{Z}$-span of W, then there is a unique minimal essential subset of W.

Fundamental Problem II: Let W be a finite subset of $\mathbb{Z}^{1\times s}$. Determine an essential set $W_0 \subseteq W$, with $|W_0|$ as small as possible.

In general the above problem is slightly more complicated since we do not have a set of vectors but a sequence with some vectors repeated.

3.5 Relations

If we are given a basic set of Brauer characters **BS** and we know how to write the basic set characters in terms of the irreducible Brauer characters, we can derive the decomposition matrix as remarked at the beginning of Section 3.3. The following notation is convenient in our situation.

Definition 3.5.1 A *relation* is a linear dependence over $\mathbb{Z}$ between generalized characters. More specificly: If $\mathbf{B}$ is any basis of $G_0(\hat{B})$, and if $\vartheta \in G_0(\hat{B})$, then the expression of ϑ in terms of $\mathbf{B}$ is called a *relation with respect to* $\mathbf{B}$. A similar notion is used for projective characters.

Example 3.5.2 The matrix in Example 3.1.3 gives the relations of the six ordinary characters not contained in **BS** in terms of this basic set. The matrix W in Example 3.3.1 gives the relations of some projectives in terms of the basic set of projectives **PS** of Example 3.1.3.

If we are given a set **BS** of ordinary irreducible characters, restricted to the p-regular classes, then in order to show that they form a basic set it suffices, as we have seen above, to express every other irreducible ordinary character, restricted to the p-regular classes, as a $\mathbb{Z}$-linear combination of the elements of **BS**. If this can be done, we have at the same time found the relations which are needed to reconstruct the whole decomposition matrix.

Sometimes the relations with respect to a basis are stored rather than the characters themselves. This is for reasons of space but also to simplify and speed up the calculation of inner products. Let **BS** be a basic set of Brauer characters with dual basis of projective atoms **PA**. Suppose furthermore, that **P** and **B** are two sets of projective characters respectively Brauer characters. Then in order to find the matrix $\langle \mathbf{P}, \mathbf{B} \rangle$ of mutual inner products, it suffices to write

$$[\mathbf{B}] = V\,[\mathbf{BS}] \tag{3.9}$$

and

$$[\mathbf{P}] = W\,[\mathbf{PA}], \tag{3.10}$$

in order to find

$$\langle \mathbf{P}, \mathbf{B} \rangle = W\,V^t. \tag{3.11}$$

Example 3.5.3 The scalar product of a Brauer character in the set

$$\{31574400, 191102976, 215547904, 219648000, 299710125, 326956500\}$$

of Example 3.1.3 with a projective of **PS** of Example 3.1.5 is given by matrix multiplication of the rows of matrix in Example 3.1.3 with the columns of the matrix of Example 3.1.5.Thus, for example, the Brauer character 191102976 has inner product $1-1-1+1-1-1+3=1$ with the projective Ψ_1 and inner product $1-1=0$ with projective Ψ_2.

3.6 Special basic sets

We keep the notation of the previous section. The situation is considerably simpler if we have basic sets of Brauer characters of a special nature.

Definition 3.6.1 A basic set **BS** of Brauer characters is called *special*, if it is of the form $\mathbf{BS} = \{\varphi_1, \ldots, \varphi_m\}$, where the φ_i are restrictions of ordinary characters of G to the p-regular conjugacy classes, i.e., $\varphi_i = \hat{\chi}_{j_i}$ for some $\chi_{j_i} \in \mathrm{Irr}(B)$.

Let $\mathbf{BS}_0$ denote a special basic set and let $\mathbf{PA}_0$ be the corresponding set of projective atoms. Then write

$$[\widehat{\mathrm{Irr}}(B)] = S\,[\mathbf{BS}_0]. \tag{3.12}$$

The matrix S thus gives the relations arising from the restrictions of the ordinary characters to the p-regular conjugacy classes with respect to the special basic set $\mathbf{BS}_0$. In the present version of MOC we always fix a special basic set and store the matrix S. In particular, MOC will only work if there exists such a special basic set.

It is not clear at all whether there is always a special basic set. In p-soluble groups there is always one by the theorem of Fong and Swan. If the ordinary irreducible characters in a p-soluble group are sorted in increasing order of their degrees, then the first maximal linear independent subsequence of characters consists of the irreducible Brauer characters. In blocks with cyclic defect group there is a special basic set by the results of Brauer and Dade. The authors have checked all blocks for all sporadic simple groups and all possible p, and found a special basic set in all cases. In [46] it is shown that there is a special basic set for groups of Lie type in non-defining characteristics under some additional hypotheses. In searching for a counterexample we would first try a group of Lie type in defining characteristic.

A special basic set has various advantages. First of all, each basic set character contains only modular constituents of a single block, and so the subset $\mathbf{BS}_0 \cap B$ is a special basic set of B. This is used to find the contribution of a generalized Brauer character to B by (2.11).

The second advantage of a special basic set is the fact that it allows to express a projective character in terms of $\mathbf{PA}_0$ without much work. Namely, let $\mathbf{P}$ denote a set of projective characters expressed in terms of the ordinary irreducible characters. That is to say we know the matrix

$$Y = \langle \mathbf{P}, \mathrm{Irr}(B) \rangle. \tag{3.13}$$

We then obtain the matrix

$$Y_0 = \langle \mathbf{P}, \mathbf{BS}_0 \rangle$$

by just deleting the columns of Y corresponding to characters in $\mathrm{Irr}(B) \setminus \mathbf{BS}_0$. Thus

$$[\mathbf{P}] = Y_0\,[\mathbf{PA}_0]. \tag{3.14}$$

Provided we have a special basic set of Brauer characters, this observation allows to restrict (in an obvious sense) the decomposition matrix to it, and then give the columns a new interpretation: These are just the PIMs expressed in the dual basis of atoms.

If one tries to calculate the decomposition matrix by constructing projective characters and expressing them in terms of ordinary irreducibles, it suffices to calculate their scalar products with the elements belonging to a special basic set. This is often an enormous saving of space and time.

Example 3.6.2 The basic set **BS** of Example 3.1.3 is special, and hence the columns of the matrix of Example 3.1.5 give the coefficients in the expression of the corresponding projectives in terms of the basis **PA** dual to **BS**.

By comparing (3.12), (3.13) and (3.14) with (3.9), (3.10) and (3.11), we have

$$Y = Y_0\, S^t.$$

This can be interpreted as finding the matrix Y if a set of projectives is given in terms of $\mathbf{PA}_0$. This will be important later on, since in the process of improving the projectives MOC uses their expressions via the matrix Y_0.

We finally specialize to the case that $\mathbf{P} = \mathbf{PS}$ is a basic set of projectives. Then, writing

$$X = \langle \mathbf{PS}, \mathrm{Irr}(B) \rangle, \qquad \text{and} \qquad X_0 = \langle \mathbf{PS}, \mathbf{BS}_0 \rangle,$$

we have

$$X = X_0\, S^t.$$

By the orthogonality relations we also have

$$[\mathbf{BS}_0] = X_0^t\, [\mathbf{BA}]. \tag{3.15}$$

If $\mathbf{B}$ is a set of Brauer characters, $[\mathbf{B}] = V\, [\mathbf{BS}_0]$, we obtain

$$[\mathbf{B}] = V X_0^t\, [\mathbf{BA}]$$

which can also be expressed as

$$\langle \mathbf{B}, \mathbf{PS} \rangle = V X_0^t.$$

This means that if $\mathbf{B}$ is given in terms of $\mathbf{BA}$, i.e., by its inner products with $\mathbf{PS}$, say $\mathbf{B} = W\, [\mathbf{BA}]$, we can express $\mathbf{B}$ in terms of $\mathbf{BS}_0$ as follows:

$$[\mathbf{B}] = W X_0^{-t}\, [\mathbf{BS}_0].$$

3.7 Triangular decomposition matrices

The solution of the Fundamental Problem I is drastically simplified if the matrix U is a lower triangular matrix. Since U is unimodular, the entries on the main diagonal are all equal to 1. The two matrices U_1 and U_2 are then unimodular triangular matrices, too. One obtains U_1, say, by iteratively subtracting columns of U from the right (compare also [80, 6.3.22] and the subsequent remarks). Closely related to this special case is the notion of wedge shape of the decomposition matrix.

Definition 3.7.1 The decomposition matrix of B has *wedge shape*, if the ordinary irreducible characters can be ordered such that the decomposition matrix has the following form:

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & * & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \\ & & & & & \\ & & * & & & \end{pmatrix},$$

where the entries above the main diagonal are all 0.

In a symmetric group every decomposition matrix has wedge shape (see e.g. [80, Theorem 6.3.60]). The same is true for the general linear and unitary groups in non-defining characteristics, as Dipper [22] and Geck [44] have shown.

We obviously have the following connection.

Lemma 3.7.2 *The decomposition matrix has wedge shape, if and only if there is a special basic set of Brauer characters* $\mathbf{BS}_0$ *and a basic set* $\mathbf{PS}$ *of projective characters such that the matrix* U *of* (3.1) *is a lower triangular matrix.* ■

Chapter 4

Data structures

In this chapter we describe the principal data structures underlying the MOC-system. We also present a new and remarkable result due to H. W. Lenstra on certain integral bases of abelian number fields. This result had been suggested by experimental evidence gained in the preparation of the MOC-system.

Based on the theory described in Chapter 3, MOC consists of a collection of FORTRAN-programs and of unformatted FORTRAN-files, which contain characters and information about the characters. The programs call the subroutine RRR to perform reading and writing of files and to do the basic operations on long integers. The files are used for exchanging information between the different programs and for recording the calculations, which have been done. The representation of the various data is the same for the external and the internal format. The programs solve very specific tasks, like tensoring two lists of Brauer characters or multiplying two matrices. MOC lacks a memory management and a language on top of it. Instead more complicated tasks are achieved by concatenating suitable programs via UNIX Bourne-shell scripts. We give an example for such a shell script in the next chapter. We begin by describing the basic data types, which the programs use, and the operations, which are performed on them.

4.1 Long integers

The fundamental data type in MOC is the long integer. A long integer n is represented as a list of coefficients with respect to the basis 10^4. If $n \geq 0$ then

$$n = \sum_{i=0}^{k} a_i 10^{4i}, \quad 0 \leq a_i \leq 9999, \quad a_k \neq 0.$$

The list of coefficients a_{k-i+1} is stored in a 1-dimensional FORTRAN-array $r[\ ]$ and 10000 is added to a_0. That is to say, $r[i] = a_{k-i+1}$, $1 \leq i \leq k$, and $r[k+1] = a_0 + 10000$. For example, if $n = 123456789$ then

$$r[1] = 1, \quad r[2] = 2345, \quad r[3] = 16789.$$

In case n is negative, we store the coefficients $-a_{k-i+1}$ and add 20000 to $-a_0$. If $n = -123456789$, then

$$r[1] = 1, \quad r[2] = 2345, \quad r[3] = 26789.$$

In order to do the basic operations on long integers, the FORTRAN programs call the arithmetic subroutine RRR, which performs addition, subtraction, multiplication and division of long integers and returns the result to the calling program. The algorithms used by RRR can be found for example in Knuth's book [88, Chapter 4.3].

4.2 Files

The data stored in memory or on file consists of numbers coded as described above and of separators which are numbers in the range between 30000 and 31000. The separators are used as labels, which indicate the sort of data, following the separator. The separator 30900 for example is followed by class functions evaluated on certain conjugacy classes.

MOC keeps track of the calculation performed for a specific group G and a specific prime p on several files. We shortly mention the four most important ones.

(a) The file `G.p` contains the following information: A special basic set $\mathbf{BS}_0$ stored under the label 30900, the matrix of relations S stored under 30550 expressing the ordinary characters in terms of $\mathbf{BS}_0$, a basic set of projectives **PS** in terms of the projective atoms dual to the special basic set, an actual basic set of Brauer characters **BS** in terms of the Brauer atoms dual to **PS**. Furthermore it contains information related to the distribution of the characters into blocks. Changing from a representation of characters in a certain basis to a representation in a different basis or writing the characters as class functions is achieved by using the formulae given in Section 3.6.

(b) The file `G.p.bras` contains a list of Brauer characters stored under the label 30500 expressed in terms of $\mathbf{BS}_0$. Whenever new Brauer characters are generated they are put onto this file.

(c) The file `G.p.proj` contains a list of projective characters stored under the label 30700 expressed in terms of the ordinary irreducible characters. Whenever new projective characters are generated they are put onto this file.

(d) The file `G.p.info` contains information concerning the calculations already performed by MOC. This can be used to give a conventional proof of the results obtained by MOC. We describe in a later section this important feature in more detail.

4.3 Integers in abelian number fields

The second basic data type deals with character values, which are algebraic integers from abelian number fields. Our representation of character values differs from those used in other systems, such as for example CAS [104]. This is motivated by the fact that Brauer character tables tend to contain more irrationalities than ordinary character tables. One of the important consequences of the new MOC-format for character values is that the basic algorithms involving Brauer characters can be reformulated as algorithms for $\mathbb{Z}$-lattices, avoiding explicit calculations in cyclotomic fields. The MOC-format also seems to give a more compact representation of a character table.

Let K be an algebraic number field of degree d over $\mathbb{Q}$ with ring of integers R. For performing arithmetic in R we choose an integral basis

$b_1, \ldots, b_d$. The elements of R are represented by the coefficients in this basis. The addition of two elements is now easily performed by adding their coefficient vectors. Next we produce the multiplication matrix of the basis elements, i.e., we store the coefficients of all products $b_i \cdot b_j$. The multiplication of two arbitrary elements of R is now achieved by matrix multiplication of the two coefficient vectors with the multiplication matrices of the basis elements. The fields K we mainly deal with have a small degree d over $\mathbb{Q}$. Therefore, we decided to store the multiplication tables explicitly, since the cost of storing is relatively small compared to the speedup we gain in performing the multiplication by the method described above.

4.4 Character tables

The character values of finite groups are sums of roots of unity and are therefore algebraic integers of abelian number fields. As described above the character values are stored as coefficient vectors corresponding to certain integral bases, which are to be described below. We shall see that this can be done in such a way that the resulting character table is still a square matrix. The new format of a character table is based upon the following considerations.

We begin by introducing some notation. If $K \subseteq L$ is a Galois extension of fields, we let $\mathcal{G}(L/K)$ denote its Galois group. Let n be a positive integer. We write $\mathbb{Q}_n$ for the n-th cyclotomic field, and $\varphi(n)$ for the degree of $\mathbb{Q}_n$ over $\mathbb{Q}$, i.e, φ is the Euler function. If f is an integer coprime to n let $*f$ denote the Galois automorphism of $\mathbb{Q}_n$ mapping an n-th root of unity to its f-th power.

Let G be a finite group and $x \in G$. Define $\mathbb{Q}(x) = \mathbb{Q}(\chi(x) \mid \chi \in \mathrm{Irr}(G))$ to be the column field of x. If x has order n then $\mathbb{Q}(x)$ is contained in $\mathbb{Q}_n$.

Definition 4.4.1 Two elements of G are called algebraically conjugate if the cyclic subgroups they generate are conjugate in G. The equivalence class of $x \in G$ with respect to this relation is called the algebraic conjugacy class of x and denoted by $Cl_a(x)$. Note that it is a union of conjugacy classes of G.

Let y be an element of $Cl_a(x)$. Then y is conjugate to x^f with f relatively prime to n. If χ is a character of G then $\chi(y) = \chi(x)^{*f}$.

Proposition 4.4.2 *Let x be an element in G of order n. Suppose the column field $\mathbb{Q}(x)$ has degree d over $\mathbb{Q}$. Then d equals the number of (ordinary) conjugacy classes in the algebraic conjugacy class $Cl_a(x)$.*

Proof. We put $Y = \{x^i \mid 1 \leq i \leq n, \gcd(i,n) = 1\}$ and write S_Y for the group of permutations of Y. The normalizer $N = N_G(\langle x \rangle)$ acts on Y by conjugation. The kernel of the corresponding homomorphism

$$\rho : N \to S_Y$$

is $C_G(x)$ and so $|\rho(N)| = |N/C_G(x)|$.

The orbits of N on Y are in one to one correspondence with the G-conjugacy classes intersecting Y non-trivially. The size of the orbits is $|N/C_G(x)|$ and the number of orbits is $\varphi(n)/|N/C_G(x)|$.

Define the monomorphism

$$\psi : \mathcal{G}(\mathbb{Q}_n/\mathbb{Q}) \to S_Y,$$

$$*k \mapsto (y \mapsto y^k)$$

and observe that $\psi(\mathcal{G}(\mathbb{Q}_n/\mathbb{Q}(x))) = \rho(N)$. This follows from the remark preceding the proposition and the fact that two elements $a, b \in G$ are conjugate, if and only if $\chi(a) = \chi(b)$ for all $\chi \in \mathrm{Irr}(G)$. Hence $d = |\mathbb{Q}(x) : \mathbb{Q}| = \varphi(n)/|\rho(N)|$ and the assertion follows. ■

We are now ready to define the MOC-character table.

Definition 4.4.3 Let $x_1, \ldots, x_r$ be representatives for the algebraic conjugacy classes of G. Let d_i denote the degree of $\mathbb{Q}(x_i)$ over $\mathbb{Q}$. For each i, we choose an integral basis $b_1(x_i), \ldots, b_{d_i}(x_i)$ of $\mathbb{Q}(x_i)$.

The columns of the MOC-table are indexed by

$$b_1(x_1), \ldots, b_{d_1}(x_1), b_1(x_2), \ldots, b_{d_2}(x_2), \ldots, b_1(x_r), \ldots, b_{d_r}(x_r)$$

and the entries of $\chi \in \mathrm{Irr}(G)$ at the columns $b_1(x_i), \ldots, b_{d_i}(x_i)$ are given by the coefficients of the decomposition of $\chi(x_i)$ in the basis $b_1(x_i), \ldots, b_{d_i}(x_i)$. By the preceding proposition the MOC-character table is a square integral matrix.

Example 4.4.4 Let A_5 be the alternating group on five letters. The ordinary character table is given by :

```
        CAS                                   MOC


              2   2  2  .  .  .
              3   1  .  1  .  .
              5   1  .  .  1  1

                 1a 2a 3a 5a 5b
             2P  1a 1a 3a 5b 5a
             3P  1a 2a 1a 5b 5a
             5P  1a 2a 3a 1a 1a              1  2  3  5  5
              2
        X.1   +   1  1  1  1  1              1  1  1  1  0
        X.2   +   3 -1  .  A *A              3 -1  0  1  1
        X.3   +   3 -1  . *A  A              3 -1  0  0 -1
        X.4   +   4  .  1 -1 -1              4  0  1 -1  0
        X.5   +   5  1 -1  .  .              5  1 -1  0  0
```

Here, $A = (1+\sqrt{5})/2$ and $*A = (1-\sqrt{5})/2$. The integral basis of $\mathbb{Q}(5A)$ in the MOC-table is $\{1, \epsilon + \epsilon^4\}$, where $\epsilon = e^{2\pi i/5}$.

We now indicate how a usual character table can be transformed into a MOC-character table and vice versa.

Definition 4.4.5 Let $\mathcal{G}(\mathbb{Q}(x_i)/\mathbb{Q}) = \{\sigma_{i,1}, \ldots, \sigma_{i,d_i}\}$, where the notation is as in Definition 4.4.3. Let A_i be the $(d_i \times d_i)$-matrix whose (j,k)-th entry is

$$\sigma_{i,j}(b_k(x_i)).$$

Let $d = \sum_{i=1}^{r} d_i$ and let A be the $(d \times d)$-block diagonal matrix whose i-th block is A_i.

Note that $(\det(A_i))^2$ equals the discriminant of $\mathbb{Q}(x_i)$ over $\mathbb{Q}$; in particular, A is invertible. With a suitable ordering of the classes, we obtain the usual character table from the MOC-character table by a matrix multiplication:

$$\text{Usual ordinary table} = \text{MOC-Table} \cdot A.$$

Note that this observation can be used to calculate the block distribution of the characters by using the MOC-table as follows. Let $\chi \in \mathrm{Irr}(G)$ and let ω_χ be the corresponding central character. Let c^i_χ denote the row of coefficients of $\omega_\chi(x_i)$ expressed in the integral basis $b_1(x_i), \ldots, b_{d_i}(x_i)$. Then

$$[\omega_\chi] = [c^1_\chi, \ldots, c^r_\chi] \cdot A.$$

Since the block matrices A_i corresponding to p-regular classes have determinant not divisible by p [105, Lemma I.(10.1), Satz I.(2.11), Korollar III.(2.10)], two characters χ and ψ are in the same p-block of G, if and only if

$$c^i_\chi \equiv c^i_\psi (\operatorname{mod} p)$$

for all i corresponding to p-regular classes. We thus have proved the stronger result on block distribution (see [47, (7.10)]), namely that two characters lie in the same p-block if and only if the values of their central characters on p-regular classes differ by elements of pR.

The MOC-system contains a data base for the subfields L of cyclotomic fields up to a certain degree. In this data base a field L is stored by the following information: The minimal n such that L is contained in the cyclotomic field $\mathbb{Q}_n$, the degree d of L over $\mathbb{Q}$, generators for the Galois group $\mathcal{L} = \mathcal{G}(\mathbb{Q}_n/L)$ and an integral basis of L. Each basis element is a sum over the elements in an orbit of $\mathcal{L}$ acting on the n-th roots of unity. Such elements are called orbit sums in the following. It is only necessary to store one representative for each orbit of $\mathcal{L}$ whose corresponding sum is a basis element.

Only recently it has been shown by H.W. Lenstra that there always exists such a basis for a given abelian extension over $\mathbb{Q}$ [95]. We are grateful to him for allowing us to reproduce this previously unpublished result. It is given in the next section.

4.5 Lenstra's theorem on integral bases

We keep the notation of the preceding section. Furthermore, we put $\zeta_m = \exp(2\pi i/m)$.

Theorem 4.5.1 (H. W. Lenstra [95]:) *Let $L/\mathbb{Q}$ be a finite abelian extension of the rational numbers. Let f be the smallest positive integer such that L is contained in the f-th cyclotomic field $\mathbb{Q}_f$ (the conductor* *of L). Finally let H be the subgroup of the Galois group $\mathcal{G}(\mathbb{Q}_f/\mathbb{Q})$ which fixes L. Then there is an integral basis of L, which consists of orbit sums of H on f-th roots of unity.*

Proof. Let p be a prime and $k \in \mathbb{N}$. We arrange the set of p^k-th roots of unity into the following matrix:

$$\begin{array}{ccccc}
1 & \zeta_p & \zeta_p^2 & \dots & \zeta_p^{p-1} \\
\zeta_{p^k} & \zeta_{p^k}\zeta_p & \zeta_{p^k}\zeta_p^2 & \dots & \zeta_{p^k}\zeta_p^{p-1} \\
\zeta_{p^k}^2 & \zeta_{p^k}^2\zeta_p & \zeta_{p^k}^2\zeta_p^2 & \dots & \zeta_{p^k}^2\zeta_p^{p-1} \\
\vdots & \vdots & \vdots & \dots & \vdots \\
\zeta_{p^k}^{p^{k-1}-1} & \zeta_{p^k}^{p^{k-1}-1}\zeta_p & \zeta_{p^k}^{p^{k-1}-1}\zeta_p^2 & \dots & \zeta_{p^k}^{p^{k-1}-1}\zeta_p^{p-1}
\end{array}$$

We introduce an equivalence relation $\sim$ on $T = \langle \zeta_{p^k} \rangle \setminus \{1\}$ as follows: For $\eta, \eta' \in T$ we put $\eta \sim \eta'$ if and only if either $\eta = \eta'$ or else $\eta^p = \eta'^p \neq 1$. The equivalence classes are the one element sets $\{\zeta_p^i\}$, $1 \leq i \leq p-1$ and the p-element sets constituted by the elements of a row different from the first of the above matrix. If C is one of the equivalence classes containing more than one element and if $\eta \in C$, then

$$\sum_{\mu \in C} \mathbb{Z}\mu = \eta\, \mathbb{Z}[\zeta_p],$$

and thus is a free $\mathbb{Z}$-module of rank $p-1$. By counting dimensions, we find

$$\mathbb{Z}[\zeta_{p^k}] = \bigoplus_{C \in T/\sim} \left(\sum_{\eta \in C} \mathbb{Z}\eta \right). \tag{4.1}$$

We now generalize these observations. Write

$$f = \prod_{p \text{ prime}} p^{k(p)}.$$

Then

$$\mathbb{Z}[\zeta_f] \cong \bigotimes_{p \text{ prime}} \mathbb{Z}[\zeta_{p^{k(p)}}]. \tag{4.2}$$

From this and the decomposition (4.1) we obtain a similar direct decomposition of $\mathbb{Z}[\zeta_f]$ which we are now going to describe. Let f_0 be the product of the different prime divisors of f. Put $S = \{\eta \in \langle\zeta_f\rangle \mid f_0 \text{ divides } |\eta|\}$. For $\eta \in S$ let $d(\eta)$ denote the largest squarefree number such that $d(\eta)^2 \mid |\eta|$. We introduce an equivalence relation $\sim$ on S as follows: For $\eta, \eta' \in S$ we put $\eta \sim \eta'$ if and only if $\eta^{d(\eta)} = {\eta'}^{d(\eta)}$. If $\eta \sim \eta'$, then $|\eta| = |\eta'|$ and so our relation is indeed symmetric. From (4.1) and (4.2), we obtain

$$\mathbb{Z}[\zeta_f] = \bigoplus_{S/\sim} \left(\sum_{\eta \in C} \mathbb{Z}\eta \right). \tag{4.3}$$

If C is an equivalence class, $\eta \in C$ and $d = d(\eta)$, then $\sum_{\mu \in C} \mathbb{Z}\mu = \eta\mathbb{Z}[\zeta_d]$, and thus the $\mathbb{Z}$-rank of $\sum_{\mu \in C} \mathbb{Z}\mu$ equals $\varphi(d)$. Thus there is a subset $B_C \subset C$ which is a $\mathbb{Z}$-basis of $\sum_{\mu \in C} \mathbb{Z}\mu$, e.g., $B_C = \{\eta\zeta_d^i \mid 0 \leq i < \varphi(d)\}$, or $B_C = \{\eta\zeta_d^i \mid \gcd(i, d) = 1\}$.

It is clear that H fixes S and permutes the equivalence classes. Suppose first that H acts fixed point freely on the set of equivalence classes of S, i.e., no non-identity element of H fixes an equivalence class. Then, by (4.3) and the subsequent remarks, it is clear that we may choose an integral basis of $\mathbb{Z}[\zeta_f]^H$, the set of integers in L, consisting of orbit sums of H on S. Thus we are done in this case.

Suppose there is some $1 \neq \sigma \in H$ and $\eta \in S$ such that $\sigma(\eta) \sim \eta$. Let $d = d(\eta)$. Then, by definition, $\sigma(\eta^d) = \sigma(\eta)^d = \eta^d$. The order of η^d is divisible by f_0, and so $\sigma(\zeta_{f_0}) = \zeta_{f_0}$. This implies $\sigma \in H_0$, where H_0 is the intersection of H with the kernel M of the natural map

$$(\mathbb{Z}/f\mathbb{Z})^* \to (\mathbb{Z}/f_0\mathbb{Z})^*.$$

The order of M is $\prod_p p^{k(p)-1}$, and the Sylow p-subgroup is cyclic, except if $p = 2$, and $k(2) \geq 3$. Let $f' = f/2^{k(2)}$ denote the odd part of f. If $k(2) \geq 3$, the Sylow 2-subgroup of M is $M_0 \times M_1$, with $M_0 = \{*a \mid a \equiv 1 (\operatorname{mod} f') \text{ and } a \equiv \pm 1 (\operatorname{mod} 2^{k(2)})\}$ and $M_1 = \{*b \mid b \equiv 1 (\operatorname{mod} 4f')\}$.

The condition that f be the conductor of L is equivalent to saying that for all p dividing f, the group H does not contain the kernel of the natural map

$$(\mathbb{Z}/f\mathbb{Z})^* \to (\mathbb{Z}/\frac{f}{p}\mathbb{Z})^*. \tag{4.4}$$

This kernel has order p and is generated by $*(1 + f/p)$.

So if p is odd or $p = 2$ and $k(2) \leq 2$, then the Sylow p-subgroup of H_0 is trivial, since otherwise $*(1 + f/p)$ would be contained in H, which is not the case by the remarks above. Thus H_0 is a 2-group, $k(2) \geq 3$ and so $H_0 \leq M_0 \times M_1$. Let $a \in \mathbb{Z}$ such that $*a \in H_0$ has order 2. Then $a \equiv 1 (\operatorname{mod} f')$ and one of the following holds:

$$a \equiv \left\{ \begin{array}{l} -1 \\ -1 + 2^{k(2)-1} \\ 1 + 2^{k(2)-1} \end{array} \right\} (\operatorname{mod} 2^{k(2)}).$$

By the remarks preceding and following (4.4), $H_0 \cap M_1 = 1$, and thus the last possibility cannot occur and we must have $H_0 = \langle a \rangle$, i.e., H_0 is a cyclic group of order 2.

So if some non-trivial element σ_0 of H fixes the equivalence class containing η, then $\sigma_0 = *a$. Furthermore, if 2 is the exact power of 2 dividing $|\eta|$, then $\sigma_0(\eta) = \eta$, and if 4 is the exact power of 2 dividing $|\eta|$, then $\sigma_0(\eta) = -\eta$. On the other hand, if $\sigma(\eta) = \eta$ or $\sigma(\eta) = -\eta$ for $\sigma \in H$ then $\sigma(\eta) \sim \eta$. Suppose that 8 divides the order of η. Write $d(\eta) = 2c$, c odd. Then $\eta \sim \sigma_0(\eta) = \eta^a$ implies $\eta^{2c} = \eta^{2ca}$, and so $\eta^{2c(a-1)} = 1$. However, $2c(a-1)$ is not divisible by 8, a contradiction.

Let $H_1 = \{*a \in H \mid a \equiv 1 (\operatorname{mod} 4)\}$. Then $H_0 \cap H_1 = 1$, and so $H = H_0 \times H_1$. We now proceed as follows. Since H_1 acts fixed point freely on the equivalence classes, we can choose an integral basis of $\mathbb{Z}[\zeta_f]^{H_1}$ consisting of orbit sums of H_1 on S. If an orbit sum contains an element from an equivalence class fixed by H_0, we consider two cases. Either the element has order divisible by 4, then H_0 acts as -1 on the corresponding $\mathbb{Z}H_1$-module. Otherwise, H_0 fixes every element in the orbit, in which case we include the corresponding orbit sum, which is now an H-orbit sum, into our basis.

The remaining orbit sums contain only elements of equivalence classes not fixed by H_0. The corresponding H-orbit sums are taken to yield the desired integral basis. ■

Chapter 5

Algorithms

In this chapter we describe the principal algorithms utilized by MOC. We first explain two algorithms for dealing with systems of integral linear equations using q-adic approximation. Then we describe the methods of integral linear programming used to attack the two fundamental problems. Furthermore we show how all MOC-computations can be documented and how these documentations are used to give a more conventional proof for the results obtained. Finally we discuss some more advanced methods which can be applied if the MOC-system does not give the complete answer.

5.1 Integral linear equations

Most of the time during a run of MOC we are dealing with the problem of expressing a character as a $\mathbb{Z}$-linear combination in terms of a basic set. In other words we have to solve a system of integral linear equations. Dixon [25] independently suggested the following way of solving this problem.

Algorithm 5.1.1 (DEC)
PROBLEM: Given $m+1$ rows of integers

$$w, b_1, b_2, \ldots, b_m \in \mathbb{Z}^{1\times n},$$

where $b_1, b_2, \ldots, b_m$ are linearly independent over $\mathbb{Z}$. If possible write w as a $\mathbb{Z}$-linear combination of these.
SOLUTION: Let T be the matrix

$$T = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Let q be a (large) prime such that $\bar{T}$, the reduction modulo q of T, has rank m. Set $v := w$, $j := 0$ and $w_0 := 0$ (zero vector). Set MAXJ to a reasonable positive integer, representing the maximum number of loops, we are willing to spend our time with.

Step 1. Check whether $\bar{v}$, the reduction modulo q of v, is in the $\mathbb{F}_q$-span of $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m$. If not, STOP, noting that w is not in the $\mathbb{Q}$-span of $b_1, b_2, \ldots, b_m$. Otherwise go to Step 2.

Step 2. Define the integers z_{ij}, $1 \leq i \leq m$ by

$$\bar{v} = \sum_{i=1}^{m} \bar{z}_{ij} \bar{b}_i, \quad \text{where} \quad -(q-1)/2 \leq z_{ij} \leq (q-1)/2,$$

and set

$$w_{j+1} := \sum_{s=0}^{j} (z_{1s}, \ldots, z_{ms}) q^s T.$$

Then every coefficient of $w - w_{j+1}$ is divisible by q^{j+1}. If $w - w_{j+1} = 0$, then STOP. Else set

$$v := (w - w_{j+1})/q^{j+1}$$

and increase j by 1. If j is larger than MAXJ then STOP, otherwise continue with Step 1.

Remark 5.1.2 Note that w is in the $\mathbb{Q}$-span of $b_1, b_2, \ldots, b_m$ if and only if

$$w = \sum_{j=0}^{\infty} (z_{1,j}, \ldots, z_{m,j}) q^j T \tag{5.1}$$

in the q-adic metric, where the integers z_{ij} are obtained by the above algorithm. Furthermore w is in the $\mathbb{Z}$-span of $b_1, b_2, \ldots, b_m$ if and only if (5.1) is finite. When the algorithm terminates because j gets larger then MAXJ, this may have three reasons.

(a) w is in the $\mathbb{Z}$-span but some coefficient is too large.

(b) w is in the $\mathbb{Q}$-span but not in the $\mathbb{Z}$-span.

(c) w is not in the $\mathbb{Q}$-span.

Our experience tells us that $q = 101$ and MAXJ $= 20$ are reasonable choices for the problems we are concerned with.

The first step in the computation of the decomposition numbers is to find a basic set for $G_0(FG)$. We start with the restrictions of the ordinary irreducible characters to the p-regular elements which form a generating set for $G_0(FG)$. We then have to derive from these a basic set of $G_0(FG)$. There are algorithms for constructing bases of $\mathbb{Z}$-lattices from generating sets, but in our situation we have the additional restriction that our basis has to consist of proper Brauer characters. This is achieved by an algorithm due to Parker, called FBA, which is based upon Algorithm 5.1.1. We describe FBA in the following general setup.

Algorithm 5.1.3 (FBA)
PROBLEM: Let X be a $\mathbb{Z}$-lattice with finite basis $\varphi_1, \ldots, \varphi_m$. Define X^+ to be the subset of X consisting of all non-negative linear combinations of the basis elements $\varphi_1, \ldots, \varphi_m$. Suppose that we are given a finite set $\mathbf{B} = \{\vartheta_1, \ldots, \vartheta_n\} \subset X^+$ generating a sublattice Y of X. Derive from $\mathbf{B}$ a basis of Y lying in X^+.

SOLUTION: Assume inductively that a subset $\tilde{B} = \{\beta_1, \ldots, \beta_{k-1}\} \subset X^+$ has been found with span equal to $\langle \vartheta_1, \ldots, \vartheta_s \rangle_{\mathbb{Z}}$, $s \leq n$, and such that $\beta_1, \ldots, \beta_{k-1}$ are linearly independent modulo some large prime q.

Step 1. If $s = n$ then STOP and output $\tilde{B}$. Otherwise take the next element $\vartheta_{s+1} \in \mathbf{B}$ and try to express it in terms of $\beta_1, \ldots, \beta_{k-1}$ using DEC with the prime q. The following cases can occur and are checked in

Step 2.

(a) If $\vartheta_{s+1} \in \langle \beta_1, \ldots, \beta_{k-1} \rangle_{\mathbb{Z}}$, increase s by 1 and go to Step 1.

(b) If $\beta_1, \ldots, \beta_{k-1}, \vartheta_{s+1}$ are linearly independent modulo q, increase s and k by 1, replace $\tilde{B}$ by $\{\beta_1, \ldots, \beta_{k-1}, \vartheta_{s+1}\}$ and go to Step 1.

(c) If $\beta_1, \ldots, \beta_{k-1}, \vartheta_{s+1}$ are linearly dependent modulo q but linearly independent over $\mathbb{Z}$, replace q by a prime ℓ such that $\beta_1, \ldots, \beta_{k-1}, \vartheta_{s+1}$ are linearly independent modulo ℓ. Replace $\tilde{B}$ by $\{\beta_1, \ldots, \beta_{k-1}, \vartheta_{s+1}\}$, increase s and k by 1 and go to Step 1.

(d) If $\beta := \vartheta_{s+1} \in \langle \beta_1, \ldots, \beta_{k-1} \rangle_{\mathbb{Q}}$ but $\beta \notin \langle \beta_1, \ldots, \beta_{k-1} \rangle_{\mathbb{Z}}$, we use the first terms of the q-adic expression for the coefficients of β determined by DEC to find the smallest possible natural number m_1 with

$$m_1 \beta = \sum_{i=1}^{k-1} z_i \beta_i,$$

where $z_i \in \mathbb{Z}$. Let $m = \min\{\gcd(m_1, z_i) \mid 1 \leq i \leq k-1\}$ and suppose $m = \gcd(m_1, z_j) > 0$. Write $m = am_1 + bz_j$, $a, b \in \mathbb{Z}$. For i different from j, let c_i be the smallest integer such that $c_i + bz_i m_1^{-1} \geq 0$ and define

$$\tilde{\beta}_j = b\beta + a\beta_j + \sum_{i \neq j}^{k-1} c_i \beta_i.$$

Then $m_1 \tilde{\beta}_j$ is contained in X^+ and hence $\tilde{\beta}_j \in X^+$, too. Replace β_j by $\tilde{\beta}_j$ in $\tilde{B}$. Increase n by 1 and put $\vartheta_{n+1} = \beta_j$. Do not increase s in this case. Go to Step 1.

Theorem 5.1.4 Algorithm 5.1.3 *terminates and finds a* $\mathbb{Z}$*-basis of* Y *contained in* X^+.

Proof. For any subset $B \subseteq \mathbf{B}$ we define the weight $w(B)$ of B to be the pair $(\mathrm{cr}(B), d(B))$. Here, $\mathrm{cr}(B)$ is the co-rank of the $\mathbb{Z}$-span of B in Y. The number $d(B)$ is the discriminant of $\langle B \rangle_{\mathbb{Z}}$, which is defined in the following way. Choose any $\mathbb{Z}$-basis $b_1, \ldots, b_s$ of $\langle B\rangle_{\mathbb{Z}}$, and let M denote the $(m \times s)$-matrix of the coefficients of $b_1, \ldots, b_s$ expressed in the basis $\varphi_1, \ldots, \varphi_m$ of X. Then $d(B) = \det MM^t$. This number is positive and independent of the basis chosen for $\langle B\rangle_{\mathbb{Z}}$.

We order the set of weights $W(\mathbf{B}) = \{w(B) \mid B \subseteq \mathbf{B}\}$ lexicographically. Observe that B generates Y if and only if $w(B)$ is minimal in $W(\mathbf{B})$.

In each of the cases occuring in Step 2 of Algorithm 5.1.3, where we change $\tilde{B}$, we decrease its weight. This is clear in cases (b) and (c), since we increase the rank of the generated lattice. In case (d) let S denote the matrix transforming $\{\beta_1, \ldots, \beta_j, \ldots, \beta_{k-1}\}$ into $\{\beta_1, \ldots, \tilde{\beta}_j, \ldots, \beta_{k-1}\}$. Then S is a diagonal matrix with diagonal entries equal to 1, except that row j of S has the entries

$$c_1 + bz_1/m_1, \ldots, c_{j-1} + bz_{j-1}/m_1, a + bz_j/m_1,$$

$$c_{j+1} + bz_{j+1}/m_1, \ldots, c_{k-1} + bz_{k-1}/m_1.$$

Thus $\det(S) = a + bz_j/m_1 = m/m_1 < 1$. This implies that the new set $\tilde{B}$ has a smaller discriminant.

Finally, by adding β_j to $\mathbf{B}$ the minimum of $W(\mathbf{B})$ remains unchanged, since the new enlarged $\mathbf{B}$ still generates the same lattice Y. This proves that the algorithm terminates. Since in each step the set $\tilde{B}$ is linearly independent over the integers, the final such set is a $\mathbb{Z}$-basis of Y. ■

5.2 Improving basic sets

Having established the major part of the theory we are now prepared to give some algorithms for dealing with the two fundamental problems stated in Sections 3.3 and 3.4. These algorithms are implemented in the MOC-system and are parts of the program IMPROVE, which uses methods of linear programming. There are some remarks concerning optimization at the end of this section.

Let us fix some notation. Let $\mathbf{PS} = \{\Phi_1, \ldots, \Phi_s\}$ be the basic set of projectives and $\mathbf{BS} = \{\varphi_1, \ldots, \varphi_s\}$ the basic set of Brauer characters. We denote by $\mathbf{PA} = \{\varphi_1^*, \ldots, \varphi_s^*\}$ and $\mathbf{BA} = \{\Phi_1^*, \ldots, \Phi_s^*\}$ the dual systems of atoms. Let **P** and **B** denote additional sets of projective respectively Brauer characters. We denote by $U = (\langle \varphi_i, \Phi_j \rangle_{ij})$ the matrix of scalar products between **BS** and **PS**.

We are now going to restate the first fundamental problem. Recall that we have to find all square integral matrices $U_1 \geq 0$ and $U_2 \geq 0$ so that $U = U_1 \cdot U_2$ subject to the conditions (3.8). We consider the equivalent problem of finding $U_1 \geq 0$ and $U_2 \geq 0$ invertible over $\mathbb{Z}$ with $U_1^{-1} \cdot U \cdot U_2^{-1} = E_s$. Operating by U_1^{-1} means changing **BS** and operating by U_2^{-1} means changing **PS**. We are going to change U according to (3.8) hoping to get E_s as result. We have the two following problems to solve:

1. Prove that φ_i is irreducible or that Φ_j is indecomposable. This is considered in § 5.2.1.

2. Improve **PS** or **BS**. We shall show that this can be done in several ways. We present a theorem in § 5.2.2 which allows to subtract irreducible characters from other ones. Then in § 5.2.3 there is an algorithm which computes characters lexicographically smaller. We finally present a criterion in § 5.2.4 for a character to be decomposable or reducible.

5.2.1 Proving indecomposability

In this paragraph we are going to establish a test for proving irreducibility of a Brauer character respectively indecomposability of a projective character. First of all we need the following

Definition 5.2.1 Let

$$\Phi = \sum_{i=1}^{s} n_i \varphi_i^* \quad \text{with} \quad n_i \geq 0$$

be a projective character decomposed into **PA**. A generalized projective character

$$\Phi' = \sum_{i=1}^{s} n_i' \varphi_i^*$$

is called a *part of* Φ if $0 \leq n_i' \leq n_i$ for all i. A similar notion is used for Brauer characters.

The projective indecomposable summands of a projective character Φ are among the parts of Φ. We are able to state the main criterion for indecomposability.

Theorem 5.2.2 (PIM-test:) *The projective Φ is indecomposable if for all parts Φ' of Φ with $\Phi' \neq \Phi$ and $\Phi' \neq 0$ there is a Brauer character $\varphi \in \mathbf{B}$ with $\langle \varphi, \Phi' \rangle < 0$ or $\langle \varphi, \Phi - \Phi' \rangle < 0$.*

Proof. If there is a φ having negative scalar product with Φ' or with $\Phi - \Phi'$ then one of Φ', $\Phi - \Phi'$ is not a genuine projective. If Φ' is a part of Φ, then so is $\Phi - \Phi'$. ■

Sometimes one can find another test for indecomposability in the literature. It is based on the fact that an ordinary character is a generalized projective character if and only if it is 0 on all p-singular classes. Let $\Phi = \sum_{i=1}^{n} a_i \chi_i$ be a projective character expressed as a sum of ordinary characters. Then we call a character $\Phi' = \sum_{i=1}^{n} a'_i \chi_i$ a *subsum of* Φ if $0 \leq a'_i \leq a_i, \Phi' \neq 0$ and $\Phi' \neq \Phi$. In the test in question we consider all subsums of Φ and check if one of them is a generalized projective. If there is no such subsum then Φ is indecomposable. Now we are going to compare the two tests.

Proposition 5.2.3 *Suppose that* **BS** *is a special basic set and that* **B** *consists of the restricted ordinary characters. Let Φ be a projective character. Then the PIM-test presented in* Theorem 5.2.2 *proves that Φ is indecomposable, if and only if there is no subsum of Φ which is a generalized projective.*

Proof. Let Ψ be an arbitrary generalized projective character and

$$\Psi = \sum_{i=1}^{n} a_i \chi_i$$

its decomposition into the ordinary irreducible characters. Then we have $\langle \hat{\chi}_i, \Psi \rangle = a_i$. Let Φ' and $\Phi'' = \Phi - \Phi'$ be parts of Φ. Then Φ' and Φ'' are generalized projective characters. Suppose there is no subsum of Φ which is a generalized projective character. Then neither Φ' nor Φ'' is a subsum. Therefore at least one of them, say Φ', has a negative coefficient in its decomposition into ordinary characters. Call this coefficient a_i.

But a_i is the scalar product of Φ' with $\hat{\chi}_i$, as we have seen above. Hence the PIM-test of Theorem 5.2.2 proves the indecomposability of Φ.

Now suppose that Φ' is a subsum of Φ and Φ' is a generalized projective character. Then $\langle \hat{\chi}_i, \Phi' \rangle \geq 0$ and $\langle \hat{\chi}_i, \Phi - \Phi' \rangle \geq 0$ for all $1 \leq i \leq n$. Because of the choice of **BS** this particular Φ' is checked by our PIM-test as well. So it can not decide whether Φ is indecomposable or not. ■

By adding more Brauer characters to **B** (for example Brauer characters got by induction) our PIM-test may find more PIMs.

Remark 5.2.4 Let $\Phi = \sum_{i=1}^{s} n_i \varphi_i^*$, $|\mathcal{B}| = m$ and denote by $v_{k1}, \ldots, v_{ks}$ the coefficients of the decomposition of the k-th character in $\mathbf{B}$, $1 \leq k \leq m$, when decomposed into **BS**. Then Φ is indecomposable if there are no $n'_1, \ldots, n'_s$ satisfying the following conditions:

$$0 \leq n'_i \leq n_i \quad \text{for all} \quad 1 \leq i \leq s,$$

$$\sum_{i=1}^{s} n'_i \leq \sum_{i=1}^{s} n_i - 1 \quad (\text{that is } \Phi' \neq \Phi),$$

$$\sum_{i=1}^{s} n'_i \geq 1 \quad (\text{that is } \Phi' \neq 0),$$

$$\sum_{j=1}^{s} -v_{kj} n'_j \leq 0 \quad \text{for all} \quad 1 \leq k \leq m \quad (\text{that is } \langle \varphi, \Phi' \rangle \geq 0 \text{ for } \varphi \in \mathbf{B}),$$

$$\sum_{j=1}^{s} v_{kj} n'_j \leq \sum_{j=1}^{s} v_{kj} \langle \varphi_j, \Phi \rangle \text{ for all } 1 \leq k \leq m \text{ (that is } \langle \varphi, \Phi - \Phi' \rangle \geq 0).$$

This system of inequations can be written in matrices as follows. There are no $n'_1, \ldots, n'_s$ satisfying $A \cdot x \leq b$ where

$$A = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 1 & \ldots & 1 \\ -1 & \ldots & -1 \\ -v_{11} & \ldots & -v_{1s} \\ \vdots & & \vdots \\ -v_{m1} & \ldots & -v_{ms} \\ v_{11} & \ldots & v_{1s} \\ \vdots & & \vdots \\ v_{m1} & \ldots & v_{ms} \end{pmatrix} \text{ and } b = \begin{pmatrix} \langle \varphi_1, \Phi \rangle \\ \vdots \\ \langle \varphi_s, \Phi \rangle \\ -1 + \sum_{j=1}^{s} \langle \varphi_j, \Phi \rangle \\ -1 \\ 0 \\ \vdots \\ 0 \\ \sum_{j=1}^{s} v_{1j} \langle \varphi_j, \Phi \rangle \\ \vdots \\ \sum_{j=1}^{s} v_{mj} \langle \varphi_j, \Phi \rangle \end{pmatrix}$$

and $x = (n'_1, \ldots, n'_s)^t$. How this system of inequalities can be solved is discussed in § 5.2.6. For the reader who is familiar with linear programming we mention that this system is dual feasible if we use a dual simplex algorithm and minimize the 0-function. Then, of course, the zero vector is a dual solution. Furthermore we observe that in each column of A the first non-zero entry is 1.

Remark 5.2.5 (i) The number of rows in A depends on $|\mathbf{B}|$. Therefore it is useful to keep $|\mathbf{B}|$ as small as possible.

(ii) By interchanging projectives and Brauer characters we can test the irreducibility of a Brauer character.

Example 5.2.6 In this chapter we are going to calculate the 5-modular table of the sporadic simple group Co_2. Its ordinary character table can be found in [19] as well as the one of the first maximal subgroup which is isomorphic to $U_6(2).2$. We make use of the sixth maximal subgroup, too. It is isomorphic to $(2_+^{1+6} \times 2^4).A_8$. The 5-modular tables of both maximal subgroups can be constructed by making use only of tensor products.

(i) In $U_6(2).2$ we have six blocks of defect 1 with the following decomposition matrices:

	Ψ_4	Ψ_3	Ψ_2	Ψ_1
1^+	1	.	.	.
616^-	.	1	.	.
8064^-	.	1	1	.
11264^+	1	.	.	1
18711_2^-	.	.	1	1

	Ψ_8	Ψ_7	Ψ_6	Ψ_5
1^-	1	.	.	.
616^+	.	1	.	.
8064^+	.	1	1	.
11264^-	1	.	.	1
18711_2^+	.	.	1	1

	Ψ_{12}	Ψ_{11}	Ψ_{10}	Ψ_9
22^+	1	.	.	.
252^-	.	1	.	.
4928^-	.	1	1	.
32768^+	1	.	.	1
37422^+	.	.	1	1

	Ψ_{16}	Ψ_{15}	Ψ_{14}	Ψ_{13}
22^-	1	.	.	.
252^+	.	1	.	.
4928^+	.	1	1	.
32768^-	1	.	.	1
37422^-	.	.	1	1

	Ψ_{20}	Ψ_{19}	Ψ_{18}	Ψ_{17}
231^+	1	.	.	.
1386^-	.	1	.	.
5544^+	1	.	1	.
14784^-	.	1	.	1
18711_1^-	.	.	1	1

	Ψ_{24}	Ψ_{23}	Ψ_{22}	Ψ_{21}
231^-	1	.	.	.
1386^+	.	1	.	.
5544^-	1	.	1	.
14784^+	.	1	.	1
18711_1^+	.	.	1	1

The remaining characters have defect 0. We write them as $\Psi_{25}, \ldots, \Psi_{59}$, where their order agrees with the one in [19].

(ii) From $(2_+^{1+6} \times 2^4).A_8$ we only need one character of defect 0 and the projectives coming from the factor group A_8. The ordinary character table of this maximal subgroup was computed by Fischer [33]. We refer the reader to his survey article [34], where he explains his methods. We obtain:

	Φ_4	Φ_3	Φ_2	Φ_1
1	1	.	.	.
14	1	1	.	.
21	.	.	1	.
56	.	1	.	1
64	.	.	1	1

	Φ_6	Φ_5
7	1	.
21_1	.	1
21_2	.	1
28	1	1

Let Φ_7 denote the defect 0 character of degree 15.

We now use MOC to calculate the 5-modular characters of the second Conway group Co_2. The Brauer characters are denoted by their degrees. The 60 ordinary characters of Co_2 split into 26 blocks. The 23 characters

$$9625_1, 9625_2, 23000, 31625_1, 31625_2, 63250, 91125_1, 91125_2, 221375,$$

$$253000, 284625, 442750, 462000, 664125_1, 664125_2, 664125_3, 853875,$$

$$1288000, 1771000_1, 1771000_2, 1992375, 2004750, 2095875$$

are of defect 0, and therefore each forms a block in its own. The characters 275, 44275, 113850, 398475 and 467775 belong to the second block. It is a block of defect 1 and the decomposition matrix can be found in [56]. It is

275	1	.	.	.
44275	.	1	.	.
113850	1	.	1	.
398475	.	1	.	1
467775	.	.	1	1

In the third block we find the characters 4025, 12650, 177100, 398475 and 558900. The decomposition matrix can be found in [56] as well:

4025	1	.	.	.
12650	.	1	.	.
177100	1	.	1	.
398475	.	1	.	1
558900	.	.	1	1

All remaining characters are in the principal block. We have to find 16 irreducible Brauer characters and PIMs. We induce the projective characters from the two maximal subgroups and call them $\Psi_1, \ldots, \Psi_{59}$ and $\Phi_1, \ldots, \Phi_{104}$. MOC finds 16 linearly independent projective characters lying in the principal block. With Lemma 3.1.4 we check that they form a basic set. They are collected in

$$\mathbf{PS} = \{\Psi_{37}, \Psi_{51}, \Psi_{46}, \Psi_{39}, \Psi_{43}, \Psi_{42}, \Psi_{38}, \Psi_{34}, \Psi_{49},$$

$$\Phi_6, \Psi_{11}, \Psi_{32}, \Psi_{31}, \Psi_{20}, \Psi_8, \Psi_4\}.$$

Next MOC finds a basic set of Brauer characters. It consists of

$$\mathbf{BS} = \{1, 23, 253, 1771, 2024, 2277, 7084, 10395_1, 31878, 37422, 129536,$$

$$184437, 212520, 239085_1, 368874, 1291059\}.$$

Furthermore we consider the following three projectives which do not decompose into **PS** with non-negative coefficients:

	Ψ_{37}	Ψ_{51}	Ψ_{46}	Ψ_{39}	Ψ_{43}	Ψ_{42}	Ψ_{38}	Ψ_{34}	Ψ_{49}	Φ_6	Ψ_{11}	Ψ_{32}	Ψ_{31}	Ψ_{20}	Ψ_8	Ψ_4
Φ_4	−1	.	.	.	.	.	1	.	.	.	.	2	.	.	.	1
Φ_5	−1	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.
Φ_7	−1	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.

We have some more Brauer characters:

	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	φ_{15}	φ_{16}
245 916	−1	1	−1	1	.	−1	−1	.	1	1	.	1	.	.	.	.
312 984	.	1	−1	−1	.	−1	1	.	−1	.	1	.	1	.	.	.
637 560	.	.	.	1	−1	−1	.	.	1	.	−1	2	.	.	1	.
1 835 008	.	−1	.	−1	.	.	1	.	−1	−1	.	.	.	1	1	1
2 072 576	.	.	.	−3	1	.	3	−1	−2	−1	.	−1	1	2	1	1

These projectives and Brauer characters define relations (see Section 3.5). With these choices of **PS** and **BS** we get the matrix U of scalar products displayed in Table 5.1. By Lemma 3.2.3 we know that Ψ_{37}, Ψ_{46} and Ψ_{39} are PIMs. We are going to show that Ψ_{43} is indecomposable. If we decompose Ψ_{43} into the atoms we get

$$[\Psi_{43}] = (0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,1) \cdot [\mathbf{PA}].$$

Therefore every part Ψ' of Ψ_{43} must be of the form

$$[\Psi'] = (0,0,0,0,0,0,0,0,0,0,0,n'_{12},0,0,0,n'_{16}) \cdot [\mathbf{PA}]$$

where $0 \leq n'_{12}, n'_{16} \leq 1$. As we should have $\Psi' \neq 0$ and $\Psi' \neq \Psi_{43}$ we get $n'_{12} = 0$ and $n'_{16} = 1$ or vice versa. But then we get a negative scalar product with 2072576. So Ψ_{43} is indecomposable. We obtain the same result for $\Psi_{42}, \Psi_{38}, \Psi_{49}$ and Ψ_{32} by using the Brauer characters 637560 and three times 2072576. Now we are going to prove that Ψ_{34} is indecomposable. We have

$$[\Psi_{34}] = (0,0,0,0,0,0,0,0,1,0,0,0,1,0,0,1) \cdot [\mathbf{PA}].$$

Table 5.1: Scalar products in the principal block of Co_2

	Ψ_{37}	Ψ_{51}	Ψ_{46}	Ψ_{39}	Ψ_{43}	Ψ_{42}	Ψ_{38}	Ψ_{34}	Ψ_{49}	Φ_6	Ψ_{11}	Ψ_{32}	Ψ_{31}	Ψ_{20}	Ψ_8	Ψ_4
1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
23	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
253	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
1771	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.
2024	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1
2277	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.
7084	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.
10395_1	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
31878	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.
37422	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.
129536	.	.	.	.	.	1	.	.	.	1	.	.	1	1	.	.
184437	.	.	.	.	1	.	.	.	.	1	.	1	.	.	.	1
212520	.	.	.	1	.	.	.	1	.	.	2	.	.	.	.	.
239085_1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
368874	.	1	.	.	.	1	.	.	1	.	.	.	.	1	1	.
1291059	1	1	.	.	1	.	1	1	.	.	1	.	.	.	1	1

Therefore every part of Ψ_{34} must be some

$$[\Psi'] = (0,0,0,0,0,0,0,0,n'_9,0,0,0,n'_{13},0,0,n'_{16}) \cdot [\mathbf{PA}]$$

with $0 \le n'_9, n'_{13}, n'_{16} \le 1$. Similarly to the preceding proof we have $1 \le n'_9 + n'_{13} + n'_{16} \le 2$. But for all those parts there is a Brauer character having negative scalar product with Ψ' or $\Psi_{34} - \Psi'$ as the following table shows:

n'_9	n'_{13}	n'_{16}	Character
0	0	1	1835008
0	1	0	312984
0	1	1	312984
1	0	0	312984
1	0	1	312984
1	1	0	1835008

Therefore, Ψ_{34} is indecomposable. We get similar results for $\Phi_6, \Psi_{11}, \Psi_{31}$

and Ψ_{20} by applying the five Brauer characters. Up to now, we know that 13 of the 16 projectives belonging to **PS** are indecomposable. In the next paragraph we are going to apply our algorithms to Ψ_{51}, Ψ_8 and Ψ_4.

5.2.2 Subtracting indecomposables

In this paragraph we are going to present an algorithm for improving basic sets by subtracting irreducible respectively indecomposable characters with the help of relations. We keep the notation introduced at the beginning of this section. To start with, we need some technical definitions.

If one of our projective characters of **PS** is known to be a PIM, we would like to have upper bounds for the number of times it can possibly be contained in any of the other projectives of the basic set. Such bounds can easily be determined.

Definition 5.2.7 Assume that Φ_j is a PIM for $j \in J$, a subset of $\{1, \ldots, s\}$. Define non-negative integers (or infinity) $m_{ij}, 1 \leq i, j \leq s$, by:

(i) If $i \notin J$, let $m_{ij} = \infty$.

(ii) If $i, j \in J$, let $m_{ij} = \delta_{ij}$.

(iii) If $i \in J$ and $j \notin J$, then

$$m_{ij} = \max\{n \in \mathbb{N}_0 \mid \langle \varphi, \Phi_j - n \cdot \Phi_i \rangle \geq 0 \text{ for all } \varphi \in \mathbf{B} \cup \mathbf{BS}\}.$$

Then obviously m_{ij} denotes how often Φ_i can possibly be contained in Φ_j. Therefore m_{ij} is called the *maximal multiplicity of* Φ_i *in* Φ_j.

Suppose that $\Phi \in \mathbf{PS}$ is a PIM. If we knew the corresponding irreducible Brauer character φ, we could of course find the multiplicity of Φ in any other projective character Ψ by calculating $\langle \varphi, \Psi \rangle$. Of course, such a situation is exceptional. In general, there will be Brauer characters which have positive scalar product with Φ, but none of them is known to be irreducible. However, each of them contains φ as a part (see Definition 5.2.1). Let ϑ be a Brauer character with positive scalar product with Φ. The idea is to consider a set of parts of ϑ containing φ.

If all these special parts, which are called bits and which are introduced in Definition 5.2.9 below, have positive scalar product with Ψ, then Φ is contained in Ψ. We can apply the same idea in a slightly more general situation.

Definition 5.2.8 Let $\Phi = n_1\varphi_1^* + \ldots + n_s\varphi_s^*$, with $n_i \geq 0$ for all i, be some generalized projective character. Then Φ is called *multiplicity free* if $n_i \in \{0, 1\}$ for all $1 \leq i \leq s$.

Definition 5.2.9 Let $\varphi \in \mathbf{BS}$, $\varphi = n_1\Phi_1^* + \ldots + n_s\Phi_s^*$ with $n_1 = \langle\varphi, \Phi_1\rangle > 0$. We are going to define a bit of φ with respect to Φ_1 if either Φ_1 is indecomposable or if Φ_1 is multiplicity free.

(i) Suppose that $\Phi_1, \ldots, \Phi_t$ are PIMs for some $1 \leq t \leq s$. Then $\varphi' = n_1'\Phi_1^* + \ldots + n_s'\Phi_s^*$ is called a *bit of φ with respect to Φ_1* if the following conditions are satisfied:

  (a) φ' is a part of φ, i.e., $0 \leq n_i' \leq n_i$ for all i,

  (b) $n_1' = 1, n_2' = \ldots = n_t' = 0, n_i' \leq m_{1i}$ for all $i > t$,

  (c) $\langle\varphi', \Psi\rangle \geq 0$ and $\langle\varphi - \varphi', \Psi\rangle \geq 0$ for all $\Psi \in \mathbf{P}$.

(ii) If Φ_1 is multiplicity free and $\Phi_{t+1}, \ldots, \Phi_s$ are indecomposable, then $\varphi' = n_1'\Phi_1^* + \ldots + n_s'\Phi_s^*$ is called a *bit of φ with respect to Φ_1* if

  (a) φ' is a part of φ, i.e., $0 \leq n_i' \leq n_i$ for all i,

  (b) $n_1' = 1$ and $n_i' \leq m_{i1}$ for all $i > t$,

  (c) $\langle\varphi', \Psi\rangle \geq 0$ and $\langle\varphi - \varphi', \Psi\rangle \geq 0$ for all $\Psi \in \mathbf{P}$.

Remark 5.2.10 Let Φ_1 be a PIM. If $|\mathbf{P}| = r$ and $v_{k1}, \ldots, v_{ks}$ are the coefficients of the k-th character in $\mathbf{P}$ when decomposed into $\mathbf{PS}, 1 \leq k \leq r$, then the conditions that φ is a bit with respect to Φ_1 can be expressed by the following system of linear inequalities $A \cdot x \leq b$ with:

$$A = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ v_{1,t+1} & \dots & v_{1,s} \\ \vdots & & \vdots \\ v_{r,t+1} & \dots & v_{r,s} \\ -v_{1,t+1} & \dots & -v_{1,s} \\ \vdots & & \vdots \\ -v_{r,t+1} & \dots & -v_{r,s} \end{pmatrix} \text{ and } b = \begin{pmatrix} \min\{n_{t+1}, m_{1,t+1}\} \\ \vdots \\ \min\{n_s, m_{1,s}\} \\ -v_{1,1} + \sum_{j=1}^{s} v_{1j}\langle\varphi, \Phi_j\rangle \\ \vdots \\ -v_{r,1} + \sum_{j=1}^{s} v_{rj}\langle\varphi, \Phi_j\rangle \\ v_{1,1} \\ \vdots \\ v_{r,1} \end{pmatrix}$$

and $x = (n'_{t+1}, \dots, n'_s)^t$. We observe that the number of rows in A and b depends on $|\mathbf{P}|$. A similar remark applies in case Φ_1 is multiplicity free.

The following two lemmas justify our definition of a bit.

Lemma 5.2.11 *If $\Phi_1, \dots, \Phi_t$ are PIMs and $\varphi \in \mathbf{BS}$ with $\langle\varphi, \Phi_1\rangle > 0$ then the irreducible Brauer character β corresponding to Φ_1 is a bit of φ with respect to Φ_1. In particular, the set of bits of φ with respect to Φ_1 is non-empty.*

Proof. As $\langle\varphi, \Phi_1\rangle > 0$ the irreducible Brauer character $\beta = \sum_{i=1}^{s} n'_i \Phi_i^*$ corresponding to Φ_1 is contained in φ. Thus it is a part of φ, hence $0 \leq n'_i \leq n_i$ for all i. Clearly, $n'_1 = 1$. All other PIMs must have scalar product 0 with β, so $n'_i = 0$ for $2 \leq i \leq t$. If $i > t$ then $n'_i = \langle\beta, \Phi_i\rangle \leq m_{1i}$. As β is a genuine Brauer character its scalar product with any projective is non-negative, therefore condition (c) is fulfilled. Thus β is a bit. ∎

Lemma 5.2.12 *Suppose that $\Phi_{t+1}, \dots, \Phi_s$ are PIMs and that $\Phi_1 \in \mathbf{PS}$ is multiplicity free. Let $\Pi_1, \dots, \Pi_s$ denote all the PIMs. We assume that they are numbered so that $\Pi_i = \Phi_i$ for $t+1 \leq i \leq s$. Furthermore, let $\Phi_1 = v_1\Pi_1 + \dots + v_s\Pi_s$. Then the following statements hold:*

(i) $v_i \in \{0, 1\}$ *for all* i.

(ii) *Let* $1 \leq k \leq s$ *with* $v_k = 1$. *If* $\varphi \in \mathbf{BS}$ *such that* $\langle \varphi, \Pi_k \rangle > 0$, *then* $\langle \varphi, \Phi_1 \rangle > 0$ *and the irreducible Brauer character* β *corresponding to* Π_k *is a bit of* φ *with respect to* Φ_1. *In particular, the set of bits of* φ *with respect to* Φ_1 *is non-empty.*

Proof.

(i) As Φ_1 is multiplicity free, its decomposition into **PA** leads to coefficients in $\{0, 1\}$. Now every PIM is a non-negative linear combination of **PA**. The result follows immediately.

(ii) Let $n'_1, \ldots, n'_s$ be the coefficients of β when decomposed into **BA**. Conditions (a) and (c) of Definition 5.2.9 are fulfilled as β is contained in φ and is a genuine Brauer character. We have

$$\langle \beta, \Phi_1 \rangle = \sum_{i=1}^{s} v_i \langle \beta, \Pi_i \rangle = v_k = 1,$$

that is $n'_1 = 1$. If $i > t$ then

$$n'_i = \langle \beta, \Phi_i \rangle = \langle \beta, \Pi_i \rangle = \delta_{ik}.$$

If $t < i \neq k$ then $\delta_{ik} = 0 \leq m_{i1}$. If $i = k$ then $n'_k = 1 = m_{k1}$ as Φ_1 is multiplicity free and contains $\Phi_k = \Pi_k$ exactly once. This completes the proof.

■

Definition 5.2.13 Let $\Phi \in \mathbf{PS}$ be indecomposable or multiplicity free. For $\varphi \in \mathbf{BS}$ with $\langle \varphi, \Phi \rangle > 0$ and a projective character $\Sigma \in \mathbf{P}$ let

$$m(\Phi, \Sigma, \varphi) = \min\{\langle \varphi', \Sigma \rangle \mid \varphi' \text{ is a bit of } \varphi \text{ corresponding to } \Phi\}.$$

Let $\Phi \in \mathbf{PS}$ be indecomposable or multiplicity free. There is at least one $\varphi \in \mathbf{BS}$ such that $\langle \varphi, \Phi \rangle > 0$. By the preceding lemmas, there is always a bit of φ with respect to Φ and so $m(\Phi, \Sigma, \varphi)$ is well defined. By our definition of bits, $m(\Phi, \Sigma, \varphi)$ is a non-negative integer. The set

$$\{m(\Phi, \Sigma, \varphi) \mid \varphi \in \mathbf{BS}, \langle \varphi, \Phi \rangle > 0\}$$

is non-empty. It is, of course, finite, since φ has only finitely many parts.

Theorem 5.2.14 *Let $\Phi \in \mathbf{PS}$ and $\Sigma \in \mathbf{P}$. If Φ is indecomposable, let*

$$z = \max\{m(\Phi, \Sigma, \varphi) \mid \varphi \in \mathbf{BS}, \langle \varphi, \Phi \rangle > 0\}.$$

If Φ is multiplicity free, let

$$z = \min\{m(\Phi, \Sigma, \varphi) \mid \varphi \in \mathbf{BS}, \langle \varphi, \Phi \rangle > 0\}.$$

Then $\Sigma - z \cdot \Phi$ is a genuine projective character.

Proof. As remarked above, z is well-defined. Suppose first, that Φ is indecomposable. Let $\{\beta_1, \ldots, \beta_s\}$ be the irreducible Brauer characters. Without loss of generality let β_1 be the one corresponding to Φ. For all $1 \leq i \leq s$ with $\langle \varphi_i, \Phi \rangle > 0$ let $z_i = m(\Phi, \Sigma, \varphi_i)$. As

$$\beta_1 \in \{\varphi' \mid \varphi' \text{ bit of } \varphi_i \text{ with respect to } \Phi\}$$

we have $\langle \beta_1, \Sigma \rangle \geq z_i$. Thus Φ is contained at least z_i times in Σ. As $z = \max z_i$, also $\Sigma - z \cdot \Phi$ is projective.

Suppose now, that Φ is multiplicity free. Let $\Phi = v_1\Pi_1 + \ldots + v_s\Pi_s$ be the decomposition of Φ into the PIMs. Then $v_i \in \{0, 1\}$ by Lemma 5.2.12(a). Now let k be such that Π_k occurs in Φ, i.e., $v_k = 1$. Let β_k be the irreducible Brauer character corresponding to Π_k. We shall show that $\langle \beta_k, \Sigma \rangle \geq z$. This will imply the result since Φ is multiplicity free.

Since $\mathbf{BS}$ is a basic set, there is some $\varphi \in \mathbf{BS}$ with $\langle \varphi, \Pi_k \rangle > 0$. Then β_k is a bit of φ corresponding to Φ by Lemma 5.2.12(b). Hence $m(\Phi, \Sigma, \varphi) \leq \langle \beta_k, \Sigma \rangle$. Since obviously $\langle \varphi, \Phi \rangle > 0$, we have $z \leq m(\Phi, \Sigma, \varphi)$. ∎

In § 5.2.6 we shall show how to solve the problem of calculating z. Changing projectives and Brauer characters give analogous results for Brauer characters.

Example 5.2.15 Let us now return to our example Co_2 mod 5. As mentioned above we still have to find three more PIMs. First we want to show that Ψ_{37} is contained in Ψ_{51}. For this purpose we have to find the irreducible Brauer character corresponding to Ψ_{37}. But this is a bit φ' of $\varphi = 1291059$. Its decomposition into the atoms is

$$[\varphi] = (1, n_2', 0, 0, n_5', 0, n_7', n_8', 0, 0, n_{11}', 0, 0, 0, n_{15}', n_{16}') \cdot [\mathbf{BA}]$$

where for all i we have $0 \leq n'_i \leq 1$. As $\Psi_{43}, \Psi_{38}, \Psi_{34}$ and Ψ_{11} are known to be indecomposable we have $n'_5 = n'_7 = n'_8 = n'_{11} = 0$. So it remains to solve the following system of linear inequalities (remember that $\langle \varphi', \Psi_{51} \rangle = n'_2$):

$$\text{Minimize } n'_2 \text{ subject to}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} n'_2 \\ n'_{15} \\ n'_{16} \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

where all $n'_i \in \mathbb{N}_0$. As this minimum equals 1 we find that $\Psi_{51} - \Psi_{37}$ is a genuine projective character. Since this projective is an atom it is indecomposable. Similarly, we show that $\Psi_8 - \Psi_{37}$ and $\Psi_4 - \Psi_{37}$ are projective. The two Brauer characters of degree 245916 and 2072576 prove that the resulting projectives are indecomposable. So we have found all PIMs. The resulting decomposition matrix as well as the table of irreducible Brauer characters is printed in the appendix.

5.2.3 Triangular matrix of scalar products

In § 5.2.2 we presented an algorithm for subtracting irreducible or multiplicity free characters. In some cases, it is possible to improve characters without knowing whether they are irreducible or not. For this algorithm we assume that the matrix U of scalar products is lower uni-triangular, i.e., $a_{ij} = \langle \varphi_i, \Phi_j \rangle = 0$ if $j > i$. Let $\{\Pi_1, \ldots, \Pi_s\}$ denote the PIMs and $\{\beta_1, \ldots, \beta_s\}$ the irreducible Brauer characters. We assume that the PIMs and **PS** are ordered consistently, i.e., that Φ_j consists of Π_j and a sum of Π_r's with $r > j$. Here we are going to describe a method which allows to subtract Π_i from some Φ_{j_0} without knowing what Π_i looks like. Suppose we know for some reason that Π_i is contained in Φ_{j_0} (see Lemma 5.2.17 below). The idea now is to subtract Φ_i from Φ_{j_0}. The possible errors due to the fact that Φ_i might not be indecomposable can easily be corrected by adding some projectives Φ_l for $l > i$ (see Theorem 5.2.18). This method produces new characters which are smaller than the older ones in the following sense. If we identify a projective Φ with the column $(\langle\varphi_1, \Phi\rangle, \ldots, \langle\varphi_s, \Phi\rangle)^t$, we can order the set of projectives lexicographically. The new characters obtained are then smaller in this lexicographic ordering.

Lemma 5.2.16 *Let $\Phi_j = \Pi_j + \sum_{r=j+1}^{s} b_{rj}\Pi_r$ be the decomposition of Φ_j into the PIMs. Then $b_{lj} \leq a_{lj} = \langle\varphi_l, \Phi_j\rangle$ for all $l = j+1, \ldots, s$.*

Proof. Let $\varphi_l = \beta_l + \mu$ for a certain Brauer character μ. Then $a_{lj} = \langle\varphi_l, \Phi_j\rangle = \langle\beta_l, \Phi_j\rangle + \langle\mu, \Phi_j\rangle = b_{lj} + \langle\mu, \Phi_j\rangle \geq b_{lj}$. ■

Lemma 5.2.17 *Let Φ be a projective character, $\Phi = \sum_{j=1}^{s} v_j\Phi_j$. Suppose that $v_i < 0$ for some i. If there exists $j_0 < i$ with $v_{j_0} > 0$ and*

$$|v_i| > \sum_{j=1, j\neq j_0, v_j>0}^{i-1} v_j a_{ij}$$

then Π_i is contained in Φ_{j_0} for a least z times, where

$$z = \lceil \frac{1}{v_{j_0}}(|v_i| - \sum_{j=1, j\neq j_0, v_j>0}^{i-1} v_j a_{ij})\rceil. \tag{5.2}$$

(Here $\lceil x\rceil$ for $x \in \mathbb{R}$ denotes the integer y with $y - 1 < x \leq y$.)

Proof. Let $\Lambda = \sum_{j=1, j\neq i, v_j\leq 0}^{s} v_j\Phi_j$. Then

$$\Phi - \Lambda + |v_i|\Phi_i = \sum_{j=1, v_j>0}^{s} v_j\Phi_j$$

(remember that $v_i < 0$). Therefore $|v_i|\Phi_i$ is contained in the right hand of this equation, hence so is $|v_i|\Pi_i$. By Lemma 5.2.16 Π_i is contained in Φ_j only for $j \leq i$ and in this case for at most a_{ij} times. So Π_i is contained in $v_{j_0}\Phi_{j_0}$ for at least

$$|v_i| - \sum_{j=1, j\neq j_0, v_j>0}^{i-1} v_j a_{ij}$$

times and therefore Π_i is contained in Φ_{j_0} at least z times. ■

Theorem 5.2.18 *Let the notation be as in* Lemma 5.2.16. *Suppose that Π_i is contained z times in Φ_{j_0} for some $i > j_0$. Then*

$$\tilde{\Phi}_{j_0} = \Phi_{j_0} - z\Phi_i + z \sum_{l=i+1}^{s} a_{li}\Phi_l \tag{5.3}$$

is a genuine projective character.

Proof. We have

$$\Phi_{j_0} - z\Phi_i + z \sum_{l=i+1}^{s} a_{li}\Phi_l$$

$$= \Phi_{j_0} - (z\Pi_i + z \sum_{l=i+1}^{s} b_{li}\Pi_l) + z \sum_{l=i+1}^{s} a_{li}\Phi_l$$

$$= (\Phi_{j_0} - z\Pi_i) + z \sum_{l=i+1}^{s} (a_{li} - b_{li})\Phi_l + z \sum_{l=i+1}^{s} b_{li}(\Phi_l - \Pi_l),$$

which by Lemma 5.2.16 is a non-negative linear combination of genuine projectives and therefore is a genuine projective character. ■

The following algorithm derived from Lemma 5.2.17 and Theorem 5.2.18 is implemented in the MOC-system. Suppose that $\mathbf{P}$ is a set of projective characters. For $\Phi \in \mathbf{P}$ write $\Phi = \sum_{j=1}^{s} v_j(\Phi)\Phi_j$.

Algorithm 5.2.19

```
For all i in 1,...,s do:
    For all j0 in 1,...,i-1 do:
        z := 0;
        For all Φ ∈ P do:
            If v_j0(Φ) ≤ 0 or v_i(Φ) ≥ 0
                    z_Φ := 0
            else
                    Calculate z_Φ according to (5.2);
            fi
            z := max{z, z_Φ};
```

od;
If $z > 0$ calculate $\tilde{\Phi}_{j_0}$ according to (5.3);
od;
od.

If, in the above algorithm, $z > 0$ for some j_0, $i > j_0$, then $\tilde{\Phi}_{j_0}$ is lexicographically smaller than Φ_{j_0} in the sense described at the beginning of this paragraph.

5.2.4 Proving decomposability

Sometimes one can show that a projective or Brauer character has to be divided into a sum of two or more characters. Let us assume that there is a projective $\Phi \in \mathbf{P}$ which has at least one negative coefficient in its decomposition into the basic set. Let $\Phi = v_1\Phi_1 + \ldots + v_s\Phi_s$ be this decomposition and assume that $v_i < 0$ for some i. Hence, Φ_i is contained in $v_1\Phi_1 + \ldots + v_{i-1}\Phi_{i-1} + v_{i+1}\Phi_{i+1} + \ldots + v_s\Phi_s$. If Φ_i is indecomposable it has to be contained in at least one $v_j\Phi_j$ for $j \neq i$. This can be checked by using an algorithm similar to the one used for calculating the m_{ij} (see Definition 5.2.7). If there is no such j then Φ_i has to be decomposable.

The easiest and most important case is that Φ_i is the sum of two indecomposable projective characters. To check this we find the two lexicographically smallest solutions $(n'_1, \ldots, n'_s)$ and $(n''_1, \ldots, n''_s)$ fulfilling the system of inequalities of Remark 5.2.4. We put $\Psi_1 = \sum n'_j\varphi^*_j$ and $\Psi_2 = \sum n''_j\varphi^*_j$. If $\Psi_1 + \Psi_2 = \Phi_i$, we are done. In two special cases it is now easy to get a better basic set:

(i) Assume that $\Psi_1 \in \mathbf{PS}$. Then $\{\Phi_1, \ldots, \Phi_{i-1}, \Psi_2, \Phi_{i+1}, \ldots, \Phi_s\}$ is a better basic set.

(ii) Assume that the matrix of scalar products is triangular. Let $\Psi_1 = \sum_{k=1}^s w_k\varphi^*_k$ and $\Psi_2 = \sum_{k=1}^s x_k\varphi^*_k$ be the decompositions into the projective atoms. Then $w_k = x_k = 0$ for all $k > i$. Without loss of generality let $w_i = 1$. As $\Phi_i = \Psi_1 + \Psi_2$ and U is unitriangular we get $x_i = 0$. In $\mathbf{PS}$ we substitute Φ_i by Ψ_1. Obviously, Ψ_1 is smaller than Φ_i. Let $l = \max\{k \mid x_k > 0\}$. Then $1 \leq l \leq i - 1$. Now we substitute Φ_l by Ψ_2 and we get the basic set required.

5.2.5 Solving the second fundamental problem

As we have seen in the last three paragraphs all algorithms depend on the size of $\mathbf{P}$ and $\mathbf{B}$. So it becomes clear that these two sets have to be kept small. First of all we remember from Section 3.4 that we only need to consider essential characters. This is made precise below.

Remark 5.2.20 We assume that there is a set E of projectives satisfying the following two conditions:

(i) $\mathbf{PS} \subseteq E \subseteq \mathbf{P}$,

(ii) For all $\Phi \in \mathbf{P}$ there exist some $v_1, \ldots, v_s, w_1, \ldots, w_r \in \mathbb{N}_0$ such that
$$\Phi = \sum_{i=1}^{s} v_i \Phi_i + \sum_{j=1}^{r} w_j \Sigma_j \tag{5.4}$$
where $E \cap (\mathbf{P} \setminus \mathbf{PS}) = \{\Sigma_1, \ldots, \Sigma_r\}$.

Then E contains "the same information" as $\mathbf{P}$ in the following sense. Suppose there is a Brauer character φ and a bit φ' of φ and a $\Sigma \in \mathbf{P} \setminus E$ with $\langle \varphi', \Sigma \rangle < 0$ or $\langle \varphi - \varphi', \Sigma \rangle < 0$. Then we have (without loss of generality $\langle \varphi', \Sigma \rangle < 0$) :

$$\Sigma = \sum_{i=1}^{s} v_i \Phi_i + \sum_{j=1}^{r} w_j \Sigma_j$$

and therefore

$$0 > \langle \varphi', \Sigma \rangle = \sum_{i=1}^{s} v_i \langle \varphi', \Phi_i \rangle + \sum_{j=1}^{r} w_j \langle \varphi', \Sigma_j \rangle.$$

This implies $\langle \varphi', \Sigma_j \rangle < 0$ for some j. So we do not need Σ.

So again we have to solve a system of linear inequalities over $\mathbb{N}_0$ which we get by subtracting $\sum_{i=1}^{s} v_i \Phi_i$ on the right hand side of equation (5.4). The problem of integer linear programming is $\mathcal{NP}$-complete. For a proof see [14]. But looking at the proof of Remark 5.2.20 we find that the conditions $v_1, \ldots, v_s, w_1, \ldots, w_r \geq 0$ are sufficient. So we are able to state the problem again, now over the field of real numbers.

In [83] one can find an polynomial algorithm for solving this problem. Nevertheless, we use an ordinary simplex algorithm for the following reasons: The solution can be constructed explicitly. So, by checking whether it satisfies the inequalities, errors due to truncation are avoided. Finally, it is quite fast. Our task is now:

Find a set E of projectives fulfilling:

(i) $\mathbf{PS} \subseteq E \subseteq \mathbf{P}$,

(ii) For all $\Phi \in \mathbf{P}$ there exists some $v_1, \ldots, v_s, w_1, \ldots, w_r \in \mathbb{R}_{\geq 0}$ with

$$\Phi = \sum_{i=1}^{s} v_i \Phi_i + \sum_{j=1}^{r} w_j \Sigma_j$$

where $E \cap (\mathbf{P} \setminus \mathbf{PS}) = \{\Sigma_1, \ldots, \Sigma_r\}$.

Here is our algorithm for solving this problem:

Algorithm 5.2.21 **Step 1.** Let $E := \mathbf{PS}, F := \mathbf{P}$.

Step 2. For all $\Phi \in F \setminus E$ do: Try to solve

$$\Phi = \sum_{i=1}^{s} v_i \Phi_i + \sum_{j=1}^{r} w_j \Sigma_j \text{ for } v_i, w_j \geq 0,$$

where $E \setminus \mathbf{PS} = \{\Sigma_1, \ldots, \Sigma_r\}$. For this purpose we decompose Φ and Σ_j into $\{\Phi_1, \ldots, \Phi_s\}$ and denote by $b_1, \ldots, b_s$ and $a_{j1}, \ldots, a_{js}$ the resulting coefficients. Then we have to find a solution $w = (w_1, \ldots, w_r)$, $v = (v_1, \ldots, v_s)$ and $v_i, w_j \in \mathbb{R}_{\geq 0}$ for the system $C \cdot x = b$ where $b = (b_1, \ldots, b_s)^t$, $A = (a_{ij})_{ij}$, $C = [A, E_s]$ and $x = [w^t, v^t]$. We multiply each equation by -1 or $+1$ so that the right hand side of any equation is non-negative. Let $\bar{C}$ and $\bar{b}$ denote the resulting coefficients. Now we introduce some artificial variables $y_1, \ldots, y_s$ which have to be greater or equal to zero and augment the system as follows:

$$\begin{array}{cccccccccc}
\bar{a}_{11}x_1 & + & \ldots & + & \bar{a}_{1,r+s}x_{r+s} & + & y_1 & & & = & \bar{b}_1 \\
\vdots & & & & \vdots & & & \ddots & & & \vdots \\
\bar{a}_{s1}x_1 & + & \ldots & + & \bar{a}_{s,r+s}x_{r+s} & & & & + \; y_s & = & \bar{b}_s
\end{array}$$

Then the system is feasible by setting $x_i = 0$ and $y_i = \bar{b}_i$. Now we minimize the function

$$f = y_1 + \ldots + y_s.$$

This function counts how many artificial variables we do need to make the system solvable. If f has a minimum equal to 0, the original system is solvable otherwise it is not. If Φ is a non-negative linear combination of elements in E throw it away, i.e. put $F := F \setminus \{\Phi\}$.

Step 3. If $F \neq E$ take a new vector $\Psi \in F \setminus E$ and put $E := E \cup \{\Psi\}$. For example, choose a vector $\Psi \in F \setminus E$ with minimal sum of coefficients. Go to 2.

Example 5.2.22 Let us go back to our example Co_2 mod 5. One useful method for generating projectives is tensoring characters of defect 0 with Brauer characters. Here we tensor 91125_1 (it is of defect 0) with 275 and restrict the resulting projective to the principal block. We call this projective Φ. Its decomposition into the first **PS** we took for Co_2 mod 5 is

$$\Phi = -\Psi_{37} + \Psi_{51}.$$

As (using the same notation as in Example 5.2.6) we have

$$\Phi_5 = \Phi + \Psi_{46} + \Psi_{42},$$

we could have thrown away Φ_5. Of course, the final result would have been the same.

5.2.6 An all-integer integer simplex algorithm

As seen in the earlier paragraphs we are given the following problem in linear programming: Let $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$. Then we have to find $\min\{c^t x\}$ subject to $x \in \mathbb{N}_0^n$ and $Ax \leq b$ where we take $\leq$ in each component. So this problem is similar to that which is solved by a simplex algorithm. The main difference is the fact that we have to find integral solutions x. So we are interested in an algorithm only using integers in order to avoid errors due to truncation. This problem was solved by R.E.Gomory in 1963 [48]. We use this Gomory algorithm to calculate the minimum taken over all bits and to check whether a character is irreducible (in this case we minimize the 0-function). The Gomory algorithm is based on the dual simplex algorithm. Here we are not going to explain the whole theory of duality and linear programming. For details see [18]. Our implementation is based on the book of Burkard [14]. Similarly to Algorithm 5.2.21, Step 2, we first augment the system of inequations by introducing some artificial variables in order to get a system of equations. We are going to explain only how we get integral solutions for the problem. We put

$$a_{00} = 0,$$

$$a_{0j} = c_j \text{ for } j = 1, \ldots, n,$$

$$a_{i0} = b_i \text{ for } i = 1, \ldots, m$$

and write these numbers in the following table:

		x_1	x_2	$\ldots$	x_n
	a_{00}	a_{01}	a_{02}	$\ldots$	a_{0n}
x_{n+1}	a_{10}	a_{11}	a_{12}	$\ldots$	a_{1n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
x_{n+m}	a_{m0}	a_{m1}	a_{m2}	$\ldots$	a_{mn}

This a shortened way of writing

$$z = \min! = \sum_{j=1}^{n} a_{0j} x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = a_{i0} \quad \text{for} \quad i = 1, \ldots, m.$$

As mentioned above, this algorithm is based on the dual simplex algorithm. We assume now that there is a solution for the dual problem such that the first non-zero element in each column a_j for $1 \leq j \leq n$ is positive. In the two algorithms presented in §§ 5.2.1 and 5.2.2 these conditions are satisfied as remarked in 5.2.4 and 5.2.10. Suppose now that the last solution the algorithm has found while optimizing has a non-integral coefficient. Then we have to exclude this point from the simplex. So we assure that all vertices of the simplex have integral coefficients. The optimal solution is found if all b_i for $i = 1, \ldots, m$ are non-negative. Otherwise one of the b_i is negative. From now on we fix this i. We are going to construct a new restriction out of the i-th row. To make the algorithm becoming all-integer the pivot element will become -1. We will repeat this until we get the optimal solution or we get a contradiction so that there is no solution. Now we are going to make the new restriction. Let a_{ij} as defined above. If $\lambda > 0$ and a is a real number then a can be written as

$$a = \lambda[\frac{a}{\lambda}] + r_a \text{ with } 0 \leq r_a < \lambda$$

where $[\cdot]$ denotes the Gaussian function. Now let $a_{i0} < 0$ for some $1 \leq i \leq m$. The corresponding restriction is

$$a_{i0} = a_{i1}x_1 + \ldots + a_{in}x_n + 1 \cdot x_{n+i}.$$

Therefore we get

$$\lambda \cdot [\frac{a_{i0}}{\lambda}] + r_0 = (\lambda \cdot [\frac{a_{i1}}{\lambda}] + r_1)x_1 + \ldots + (\lambda \cdot [\frac{a_{in}}{\lambda}] + r_n)x_n + (\lambda \cdot [\frac{1}{\lambda}] + r_{n+i})x_{n+i}$$

and then

$$\sum_{j=1}^{n} r_j x_j + r_{n+i}x_{n+i} = r_0 + \lambda\{[\frac{a_{i0}}{\lambda}] - \sum_{j=1}^{n}[\frac{a_{ij}}{\lambda}]x_j - [\frac{1}{\lambda}]x_{n+i}\}.$$

Since $r_j \geq 0$ and $x_j \geq 0$ for $j = 1, \ldots, n, n+i$ the left hand of this equation is non-negative. Also, the term in brackets is integral. Let

$$\bar{x} = [\frac{a_{i0}}{\lambda}] - \sum_{j=1}^{n}[\frac{a_{ij}}{\lambda}]x_j - [\frac{1}{\lambda}]x_{n+i}.$$

If $\bar{x} < 0$, then $\bar{x} \leq -1$, hence $\lambda\bar{x} \leq -\lambda$. As $r_0 < \lambda$ we would have $r_0 + \lambda\bar{x}$ being negative, a contradiction. In this algorithm we are going to choose $\lambda > 1$. Then $[\frac{1}{\lambda}] = 0$ and we get the restriction

$$\bar{x} = [\frac{a_{i0}}{\lambda}] - \sum_{j=1}^{n}[\frac{a_{ij}}{\lambda}]x_j \geq 0.$$

From this construction we get that every integral point of the given problem satisfies this restriction. We are going to choose a λ such that the pivot element will become -1. Hence the system will remain integral after pivoting. In the dual simplex algorithm the column s containing the pivot element is determined by

$$\frac{a_{0s}}{[\frac{a_{is}}{\lambda}]} = \max\{\frac{a_{0j}}{[\frac{a_{ij}}{\lambda}]} \mid a_{ij} < 0, j = 1, \ldots, n\}.$$

If there is no such $a_{ij} < 0$ then the system is not solvable. As we are going to choose λ such that $[\frac{a_{is}}{\lambda}] = -1$, we get

$$-a_{0s} = \max\{\frac{a_{0j}}{[\frac{a_{ij}}{\lambda}]} \mid a_{ij} < 0, j = 1, \ldots, n\}.$$

But $[\frac{a_{ij}}{\lambda}]$ is an integer, less or equal to -1. If we put

$$\mu_j = [\frac{a_{ij}}{\lambda}]$$

we have

$$-a_{0s} \geq \frac{a_{0j}}{\mu_j} \geq -a_{0j},$$

hence s is determined by

$$-a_{0s} = \max\{-a_{0j} \mid a_{ij} < 0, j = 1, \ldots, n\}.$$

This is equivalent to

$$a_{0s} = \min\{a_{0j} \mid a_{ij} < 0, j = 1, \ldots, n\}.$$

If s is not determined uniquely consider the next rows of the table to make a decision. We determine for $k = 1, 2 \ldots$

$$a_{ks} = \min\{a_{kj} \mid a_{ij} < 0, a_{k-1,j} \text{ minimal for } a_{ij} < 0\}$$

until the minimum is unique. So we achieve that for all j with $a_{ij} < 0$ the first element of the vector $a_s - a_j$ is negative. We are now going to examine how to choose λ best. After pivoting we get a new value

$$\bar{a}_{00} = a_{00} + [\frac{a_{i0}}{\lambda}]a_{0s}$$

for the function to be minimized. In order to decrease the value as much as possible we have to choose λ as minimal as possible. But we have to make sure that the first non-zero element $\bar{a}_{kj}$ of the columns $j = 1, \ldots, n$ with $j \neq s$ is

$$\bar{a}_{kj} = a_{kj} + [\frac{a_{ij}}{\lambda}]a_{ks} > 0.$$

Let i_0 be the minimum of $i(j)$ and $i(s)$ where $i(j)$ is first index for which $a_{i(j),j} \neq 0$ (and therefore $a_{i(j),j} > 0$). Similarly, let $i(s)$ be the first index of rows with $a_{i(s),s} > 0$. Then $\bar{a}_{i_0 j}$ becomes the first non-zero element of column j. If $a_{ij} \geq 0$ then we have $\bar{a}_{i_0 j} > 0$ for all possible λ as we either have $a_{i_0 j} = 0$ and $a_{i_0 s} > 0$ or $a_{i_0 j} > 0$ and $a_{i_0 s} \geq 0$. But if we have $a_{ij} < 0$ and $a_{i_0 j} > 0$ then we get a new restriction for λ resulting from the equation determing $\bar{a}_{kj}$. As $0 \leq a_{is} \leq a_{i0}$ we get

$$-1 \geq \mu_j > -\frac{a_{i_0 j}}{a_{i_0 s}}.$$

As λ has to be chosen as small as possible and $a_{ij} < 0$ we have to choose μ_j as small as possible, too. Therefore we take μ_j to be the smallest integer greater than $-\frac{a_{i_0 j}}{a_{i_0 s}}$. Now for $j \in J = \{j \mid a_{ij} < 0\}$ we put:

$$\lambda_j = \begin{cases} \frac{a_{ij}}{\mu_j} \text{ if } a_{i_0 s} > 0 \\ 0 \text{ otherwise} \end{cases}.$$

Then

$$\lambda = \max\{\lambda_j \mid j \in J\}$$

satisfies all conditions, i.e. $[\frac{a_{is}}{\lambda}] = -1$, the value of the function being optimized decreases as much as possible and after pivoting in all columns the first non-zero element is positive.

Algorithm 5.2.23 Gomory's all-integer integer algorithm (see [14]) for solving the problem

$$\sum_{j=1}^{n} a_{0j} x_j = \min!$$

subject to $Ax \leq a_0$ for $x_j \in \mathbb{N}_0$ and $1 \leq j \leq n$. We assume that the problem is dual feasible. Then:

1. If for all $i = 1, \ldots, m$
$$a_{i0} \in \mathbb{N}_0$$
then the optimal solution is found. Stop. Otherwise go to Step 2.

2. Let
$$\tilde{r} := \min\{i \mid a_{i0} \notin \mathbb{N}_0, i = 1, \ldots, m\}.$$

3. If $a_{\tilde{r}j} \geq 0$ for all $j = 1, \ldots, n$ then the problem is not solvable. Stop. Otherwise let
$$J := \{j \mid j > 0 \text{ and } a_{\tilde{r}j} < 0\}.$$

4. Determine s to be the index of the pivoting column by
$$a_{0s} := \min\{a_{0j} \mid j \in J\}.$$
If s is not determined uniquely repeat for $i = 1, 2, \ldots$
$$a_{is} := \min\{a_{ij} \mid j \in J \text{ and } a_{i-1,j} \text{ minimal }\}$$
until the minimum is determined uniquely.

5. For $j \in J$ let $i(j)$ be the first index of rows such that $a_{i(j),j} \neq 0$. Determine μ_j to be the smallest integer for which
$$a_{i(j),j} + \mu_j a_{i(j),s} > 0.$$
If $a_{i(j),s} = 0$ put $\mu_j := -\infty$.

6. For $j \in J$ let
$$\lambda_j := \begin{cases} 0 \text{ if } \mu_j = -\infty \\ \frac{a_{\tilde{r}j}}{\mu_j}, \text{ otherwise} \end{cases}$$
and let
$$\lambda := \max\{\lambda_j \mid j \in J\}.$$
If $\lambda = 1$, replace λ by $1 + \varepsilon$ where ε is some arbitrary little number.

7. For $j = 0, 1, \ldots, n$ put
$$a_{m+1,j} := \lceil \frac{a_{\tilde{r}j}}{\lambda} \rceil$$
and add $a_{m+1,0}, a_{m+1,1}, \ldots, a_{m+1,n}$ as $(m+1)$st row to the table. Let $r := m := m + 1$.

8. Put
$$\bar{a}_{rj} := -a_{rj} \text{ for } j = 0, 1, \ldots, n; j \neq s,$$
$$\bar{a}_{ij} := a_{ij} + a_{is}a_{rj}$$
$$\text{for } i = 0, 1, \ldots, m-1, j = 0, 1, \ldots, s-1, s+1, \ldots, n.$$

9. Let for $i = 0, 1, \ldots, m; j = 0, 1, \ldots, n; j \neq s$
$$a_{ij} := \bar{a}_{ij}$$
and go to 1.

5.3 Automatic proofs

We once more emphasize the fact that MOC does **not** provide an algorithm for calculating the irreducible modular characters for a given finite group. No such algorithm is yet known. Rather MOC provides a collection of algorithms for various purposes to support the calculation of some decomposition numbers and irreducible Brauer characters.

Finding the modular characters of a finite group is a kind of problem where a result can be checked with much less effort than was needed to obtain it. We have to produce a very large number of Brauer characters and projective characters to begin with. If the irreducible respectively indecomposable characters happen to be among those of our pool, then we would find them by means of the scalar product (see Proposition 3.3.2).

The methods described in Section 2.5 for constructing characters do not allow to select good candidates beforehand. Usually most of the characters generated are not needed at all. Only after the construction can one recognise those which really contain the information wanted. The other characters constructed during the session can be thrown away. For the final proof, which can usually be given in a conventional form, it suffices, of course, to keep the "good guys".

Every run of MOC is documented on an info-file corresponding to the pair (G, p). Some additional information, for example output of certain programs is stored on this file, which is called `G.p.info`. This method of automatic documentation is very much supported by the UNIX operating system. As already mentioned, MOC consists of a collection of FORTRAN programs and UNIX Bourne-shell scripts. The FORTRAN programs serve some well defined purposes such as, for example, matrix multiplication or solving integral linear equations. Transfer of data is via files, and the files are moved by shell scripts according to the needs of the programs.

Let us give an example of such a shell script. In MOC, the actual basic set **PS** of projective characters is stored on the file `G.p` under the label 30700 as a matrix X_0, expressing **PS** in terms of the projective atoms $\mathbf{PA}_0$ corresponding to the special basic set $\mathbf{BS}_0$. By equation (3.14) we have $X_0 = \langle \mathbf{PS}, \mathbf{BS}_0 \rangle$. The following shell script, called `bsinba` transposes the matrix X_0 stored under label 30700 on the file `G.p`. By equation (3.15) this expresses the special basic set $\mathbf{BS}_0$ in terms of the Brauer atoms **BA** corresponding to **PS**. Since the FORTRAN program `transp` to transpose a matrix expects the matrix under label 30900, we have to `merge` it first from 30700 to 30900. The shell script `get` copies the file `G.p` to the current working directory, which for safety reasons should always be different from the directory containing the data-files. The first argument of the following shell script is the prime p, the second the name of the group G.

```
set -x
# bsinba
rm -f t1 t2 t3 t4
get $2.$1 || exit
cp $2.$1 t1
cp t1 t2
merge <<%
30900
0
30700
%
rm -f t2
mv t3 t1
transp
mv t2 t3
rm -f t1 t2
```

The shell script expressing the Brauer atoms in terms of the special basic set $\mathbf{BS}_0$ is called bainbs. By equation (3.15) it has to invert the matrix X_0^t.

```
set -x
#
# bainbs
#
# Expresses the Brauer atoms corresponding to
# the actual  basic set of projectives, which
# are stored in $2.$1 under the label 30700,
# in terms of the special basic set.
# The result is put onto file t3.
#
# ``bsinba'' is the program inverse to ``bainbs''
#
bsinba $1 $2
mv t3 hilf
#
# ``invers'' calculates the inverse of a matrix
#
invers hilf
rm t1 t2
mv hilf.inv t3
```

The following shell script calculates the set of Brauer atoms **BA** as class functions. For this purpose it has to multiply the matrix X_0^{-t} with $[\mathbf{BS}_0]$, the latter being stored on G.p under the label 30900.

```
set -x
#
# bratoms
#
# This program gives the $1-modular
# Brauer atoms of group $2 corresponding
# to the actual basic set of projective
# characters as class functions.
# The answer is stored in $2.$1.bat.
#
```

```
bainbs $1 $2
mv t3 t2
get $2.$1 1
matmul2
mv t3 $2.$1.bat
rm -f $2.$1 t1 t2 t4
```

We now give an example of an info-file which documents the session for determining the Brauer characters for the Mathieu group M_{11} modulo 5.

```
Fri Mar 20 11:25:16 MET 1992
/home/euterpe/hiss/mocha/proggy/prp
pst run
fct run

projectives no     1 -    5 obtained by:
defect 0 characters

relations stored under 30550

Brauer characters no     1 -   10 obtained by:
ordinary characters restricted to p-regular classes

Fri Mar 20 11:25:56 MET 1992
/home/euterpe/hiss/mocha/proggy/defzo

projectives no     6 -   55 obtained by:
tensoring ordinaries with defect 0 characters
option:1:all ordinaries tensor all defect 0 characters

Fri Mar 20 11:26:17 MET 1992
/home/euterpe/hiss/mocha/proggy/basepro
input label (i5)
insert depth of search (i3), e.g. 10
input block-number (i3)
number of basic characters found is   4
numbas= 10011 10012 10010 10007
```

```
Fri Mar 20 11:27:21 MET 1992
/home/euterpe/hiss/mocha/proggy/protest
projective  nr    11 in block   1 is indecomposable,
  because it is an atom
projective  nr    12 in block   1 is indecomposable,
  because it is an atom
projective  nr    10 in block   1 is indecomposable,
  because it is an atom
projective  nr     7 in block   1 is indecomposable,
  because it is an atom
```

The info-file serves two principal purposes. It can be used to repeat the calculations for a particular group and prime. This may be necessary if some information is lost by accident or if the files get too large and immovable. In a second run one can usually proceed much faster, because one already knows which of the steps have been redundant.

The second purpose is of course the reconstruction of those characters which are used in the proof. Every character generated is supplied with an identification number. In particular, every irreducible Brauer character or indecomposable projective character has its generating number, which can be used to trace back to its origin with the help of the info-file. We even have programs which do the trace-back automatically, at least if the character was obtained by some standard constructions. This gives the method of automatic proofs. In [56] about two thirds of the proofs were obtained in this fashion. If such a proof is written down in conventional form everybody can reconstruct the characters by the description we have given and thus check the proof by hand. An example of such a proof is given in Chapter 6, where we have collected all the information necessary to determine the decomposition numbers modulo 7 of Conway's largest group Co_1.

5.4 Extensions

There are some features which we would like MOC to have, but which are still not implemented. The first and most important one is the additional information one obtains from duality and group automorphisms. For instance, a real valued ordinary character must be contained in a

PIM and its dual with equal multiplicity. This simple observation and analogous statements in the case of group automorphisms can be used to improve the algorithms described in Section 5.2 considerably. This is one of the next things we are planning to add to MOC.

The additional information available in characteristic 2 by Fong's lemma (see 5.5.3 below) has not been implemented yet.

There are examples such as the Rudvalis group in characteristic 5 which seem to be very hard to settle with the current methods of MOC. Here, no improvement of the characters as described in Section 5.2 is possible, but still there seems to be enough information to rule out most of the cases. Here, it really is desirable to have a program which tries to solve Fundamental Problem I in its full generality. We plan to do some experiments with these factorization problems.

Finally, we would very much like to have a general trace-back program for writing down more complicated proofs. The current versions cannot cope with the numerous possibilities for generating and improving characters.

5.5 Advanced methods

As remarked in the introduction, MOC uses only elementary methods to calculate decomposition numbers. In large examples, however, these methods are not sufficient to find all of the irreducible Brauer characters.

One of the reasons for this is the fact that there may be proper ordinary characters which vanish on all p-singular classes and thus are virtual projective characters, but which are not characters of projective modules. This means that in the decomposition matrix there are two columns, corresponding to PIMs Φ and Ψ, such that one is contained in the other, in other words, $\Phi - \Psi$, say, is a proper ordinary character. This happens for example in the case of the principal 3-block of the second Janko group J_2, which has the decomposition matrix displayed in Table 5.2 (for a proof see [55]). In this example $\Phi_2 - \Phi_6$ is a proper ordinary character. Table 5.3 gives the character table of a basic set of Brauer characters. The table is given in the MOC-character table format. We use the integral bases $\{1, (-1+\sqrt{5})/2\}$ for the occuring irrationalities.

Let φ denote the irreducible Brauer character corresponding to Φ.

Table 5.2: The 3-modular decomposition of J_2

	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8
1	1	.	.	.	.	.	.	.
14	1	1	.	.	.	.	.	.
14	1	.	1	.	.	.	.	.
21	.	.	.	1	.	.	.	.
21	.	.	.	.	1	.	.	.
70	.	1	.	.	.	1	.	.
70	.	.	1	.	.	.	1	.
160	1	1	1	.	.	.	.	1
175	.	.	.	1	1	.	.	1
224	.	1	.	.	1	1	.	1
224	.	.	1	1	.	.	1	1
300	1	2	2	.	.	1	1	1
336	2	1	1	1	1	.	.	2

The proof that Ψ is not a direct summand of the projective with character Φ is usually achieved by giving a lower bound for the multiplicity of φ in the reduction modulo p of some ordinary character.

In the above example, φ is an irreducible Brauer character of degree 13, and to show that $\Phi_2 - \Phi_6$ is not projective, is the same as showing that φ is contained in the reduction modulo 3 of the first ordinary character of degree 70. By inspecting the table of Brauer characters 5.3 we see that

$$169 = 13_1 \otimes 13_1 = 1 - 13_1 + 21_1 + 70_1 + 90,$$

and thus 13_1 must be contained in 70_1. This example is typical insofar as we have used tensor products of Brauer characters to obtain the bounds. The larger the group and hence the degrees of the irreducible Brauer characters, the weaker gets this method. Sometimes it helps to distinguish cases as indicated in Section 3.3 with the help of Brauer characters obtained by tensor products.

If all this fails, there are more advanced methods available, which make use of structure theory of G-modules, which we are now going to

Table 5.3: Some 3-modular characters of J_2

$1A$	$2A$	$2A$	$4A$	$5A$	$5A$	$5C$	$5C$	$7A$	$8A$	$10A$	$10A$	$10C$	$10C$
1	1	1	1	1	0	1	0	1	1	1	0	1	0
13	−3	1	1	2	3	1	1	−1	−1	−1	1	0	1
13	−3	1	1	−1	−3	0	−1	−1	−1	−2	−1	−1	−1
21	5	−3	1	4	1	2	2	0	−1	−1	−1	0	0
21	5	−3	1	3	−1	0	−2	0	−1	0	1	0	0
70	−10	−2	2	5	5	0	0	0	0	0	−1	0	0
70	−10	−2	2	0	−5	0	0	0	0	1	1	0	0
133	5	1	−3	−7	0	−2	0	0	1	1	0	0	0
36	4	0	4	−4	0	1	0	1	0	0	0	−1	0
90	10	6	−2	5	0	0	0	−1	0	1	0	0	0
63	15	−1	3	3	0	−2	0	0	1	−1	0	0	0
225	−15	5	−3	0	0	0	0	1	−1	0	0	0	0
189	−3	−3	−3	3	3	2	1	0	1	0	1	−1	−1
189	−3	−3	−3	0	−3	1	−1	0	1	−1	−1	0	1
169	9	1	1	13	3	2	1	1	1	2	−3	1	−1

describe. For every such method we also give an example where it can be applied. In fact, all the tricks to be described in the following have come to mind only when we were trying to solve the corresponding problems.

Let S be one of the rings $\{K, R, F\}$ of our p-modular system for G. We only consider SG-modules X, which are free and finitely generated as S-modules. Let X and Y be two such SG-modules. We set

$$[X, Y] = \operatorname{rank}_S \operatorname{Hom}_{SG}(X, Y).$$

The SG-module dual to X is denoted by X^*.

5.5.1 Associativity of the tensor product

Let X, Y and Z denote three SG-modules. Then there is a canonical isomorphism of SG-modules:

$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z).$$

This can sometimes be used if the dimensions of the modules involved in the tensor product are small.

5.5.2 Adjointness

Let X, Y and Z denote three SG-modules. We then have (see [93, Corollary II.6.9])

$$[X \otimes Y, Z] = [X, Y^* \otimes Z].$$

This has been used in the final state of our example of the Conway group modulo 7 (see Section 6.2), and also in the proof for the Janko group J_2 modulo 5 [55]. This method can be used either to show that some composition factor must occur in a certain tensor product (J_2 modulo 5) or to show that it cannot occur (Co_1 modulo 7).

It is worth noting that via tensor products one can transfer information between different blocks with the help of this adjointness property. This is particularly helpful if some of the blocks involved are of defect 1, since for those one can write down all the indecomposables, if the Brauer tree is known. An example of a successful application is provided by the second Janko group J_2 modulo 3.

5.5.3 Nakayama relations

Let H be a subgroup of G and let X be an SG-module, Y an SH module. Then we have the Nakayama relations (or Frobenius reciprocity) (see [93, Corollary II.1.4])

$$[X, Y^G] = [X_H, Y], \quad \text{and} \quad [Y^G, X] = [Y, X_H].$$

This has been applied for instance in the calculation of the modular characters for the Tits group ([50]).

5.5.4 Self duality

If an FG-module X is self dual, i.e., $X \cong X^*$, then its socle series is the dual of its Loewy series (see [93, Lemma I.8.4(i)]). In particular, any self dual composition factor M in the head of X must also occur in the socle. Hence such an M either is twice a composition factor of X or else it is a direct summand. If the latter is not the case, X has a non-trivial endomorphism.

The following lemma, which follows from the above remarks, is often quite useful.

Lemma 5.5.1 *Let X be a self dual FG-module with $[X, X] = 1$. If every composition factor of X is self dual, then X is irreducible.* □

5.5.5 Trivial source modules

The methods described above are very powerful in connection with the theory of modules with a trivial source. These are, by definition, indecomposable direct summands of permutation modules over F or R. Every indecomposable FG-module M with trivial source is liftable to a uniquely determined trivial source RG-module X (see [93, Section II.12]). This means that M is isomorphic to the FG-module $X \otimes_R FG$. Furthermore, $[M, M] = [X, X]$.

Any direct summand of the tensor product of two trivial source modules is again a trivial source module. If H is a subgroup of G and Y a trivial source module of H, than every direct summand of Y^G is a trivial source module.

If B is a block of G containing only real valued ordinary characters, then B is called *strongly real*. In a strongly real block every simple FG-module is self dual. This follows from the surjectivity of the decomposition homomorphism: every irreducible Brauer character in such a block is real valued and thus corresponds to a self dual simple module.

Let M be a trivial source FG-module with trivial source lift X. Suppose that the ordinary character of X is irreducible and that M is contained in a strongly real block. Then M is irreducible by Lemma 5.5.1. This applies in particular to permutation modules of doubly transitive permutation representations on n points if p does not divide n. If $1 + \chi$ is the ordinary character of a doubly transitive permutation representation, then χ is irreducible modulo p if χ is contained in a strongly real block and if p does not divide $1+\chi(1)$. The symplectic group $S_6(2)$ has a doubly transitive permutation representation of degree 28, and, since the ordinary character table of $S_6(2)$ has only rational entries, the non-trivial constituent of the permutation character remains irreducible modulo 3.

5.5.6 Lattices

An RG-module X, which is free as R-module, is also called an *RG-lattice*. A sublattice Y of X is called *pure*, if the factor module X/Y again is a lattice. The following result is more or less folclore.

Lemma 5.5.2 *Let X be an RG-lattice with ordinary character $\chi+\psi$. Then there exists a pure sublattice $Y \le X$ such that the character of Y is χ and the character of X/Y is ψ.*

Proof. See [93, Theorem I.17.3]. ■

This has turned out to very powerful in the theory of blocks with a cyclic defect group. But it has also found applications for non-cyclic defect groups, for example in the proof of Theorem 1 of [50].

5.5.7 Fong's lemma

Lemma 5.5.3 (Fong's lemma:) *Let $p = 2$. Then every self dual non-trivial simple FG-module has even dimension.*

Proof. Such a module carries a non-degenerate G-invariant symplectic form (see [69, Theorem VII.8.13]). ■

Let $\chi \in \mathrm{Irr}(G)$ be real-valued. Then it easily follows from Fong's lemma that $d_{\chi 1} \equiv \chi(1) (\bmod 2)$.

For example, consider the Mathieu group M_{11} modulo 2. There are two characters of defect 0, namely χ_6 and χ_7 (in Atlas notation). If we define

$$\Psi_1 := \chi_7 \otimes \chi_6,$$

$$\Psi_2 := \chi_6 \otimes \chi_6,$$

$$\Psi_3 := \chi_2 \otimes \chi_6,$$

we obtain the following basic set of projective characters for the principal 2-block of M_{11}.

	Ψ_1	Ψ_2	Ψ_3
1	1	.	.
10	.	1	.
10	.	1	.
10	.	1	.
11	1	1	.
44	1	1	1
45	2	1	1
55	2	2	1

Fong's lemma shows that Ψ_3 is contained in Ψ_1.

5.5.8 Theorem of Benson and Carlson

Theorem 5.5.4 (Benson-Carlson:) *Let M, N be indecomposable FG-modules such that $p \mid \dim_k M$. Then every indecomposable direct summand of $M \otimes N$ has dimension divisible by p.*

Proof. See [8, Vol. I,Theorem 3.1.9], ■

This is applied, for example, in [59, p. 111].

5.5.9 Blocks with cyclic defect groups

In a block with a cyclic defect group one can describe all indecomposable FG-modules, once the planar embedded Brauer tree is known (See [1, Chapter V]).

This information is usually easier to obtain than that for a block with a non-cyclic defect group. It is very powerful in connection with the Nakayama relations (where we assume that some block for a subgroup with a cyclic defect group is known) and, if some blocks with a cyclic defect group for G are known, in connection with the adjointness or the theorem of Benson-Carlson.

This has been applied, for example, for some blocks of the Monster group in characteristic 5 (see [56, p. 454f]), and also in the Rudvalis group modulo 3 [54].

5.5.10 Webb's theorem

Let M be an indecomposable FG-module. The heart of M is the FG-module $H(M) = \mathrm{rad}(M)/\mathrm{soc}(M)$, where $\mathrm{rad}(M)$ denotes the intersection of all maximal submodules of M and $\mathrm{soc}(M)$ is the sum of all simple submodules. It is not difficult to see that if M is self dual, the same is true for the heart of M. By classifying the possible Auslander-Reiten quivers for a finite-dimensional group algebra, Webb has shown that the heart of a projective indecomposable module of FG has at most 4 indecomposable direct summands. This result has been extended by Bessenrodt.

Theorem 5.5.5 (Webb, Bessenrodt:) *Let P be a projective indecomposable module of FG. Then $H(P)$ has at most 3 indecomposable direct summands.*

Proof. See [10, Corollary 1.3]. ■

We can apply this in some situations as follows.

Lemma 5.5.6 *Let B be a strongly real p-block of G. Let Φ be a PIM of B such that $\langle \Phi, \Phi \rangle < 4$ and*

$$\sum_{\Psi \in \mathrm{IPr}(B)} \langle \Phi, \Psi \rangle \geq 6. \tag{5.5}$$

Then there exists some $\Theta \in \mathrm{IPr}(B)$, $\Theta \neq \Phi$ such that $\langle \Phi, \Theta \rangle \geq 2$.

Proof. Let P denote the projective indecomposable FG-module with character Φ. By assumption (5.5), the heart $H(P)$ of P has at least 4 composition factors. By Bessenrodt's extension of Webb's theorem, $H(P)$ has at most 3 indecomposable direct summands, hence it cannot be semi-simple. It follows that $H(P)$ contains some composition factor with multiplicity at least 2. Since $\langle \Phi, \Phi \rangle < 4$, this must be a simple module not isomorphic to $P/\mathrm{rad}(P)$ and the assertion follows. ■

For example, this lemma can be applied in the case of the Rudvalis group modulo 3 [54]. We finally proof a lemma in the same spirit as the one above.

Lemma 5.5.7 *Suppose that G has no factor group of order p. Suppose also, that the principal block of G is strongly real. Let Φ denote the character of the projective cover P of the trivial FG-module. If $\langle \Phi, \Phi \rangle > 2$, then there exists some $\Theta \in \mathrm{IPr}(B)$, $\Theta \neq \Phi$ such that $\langle \Phi, \Theta \rangle \geq 2$.*

Proof. If the trivial FG-module is in the second Loewy layer of P, then G has some factor group of order p (see [93, Corollary I.10.13]). It follows that $H(P)$ is not semi-simple and that it must contain some non-trivial composition factor with multiplicity at least 2. ∎

Chapter 6

Calculating the 7-modular decomposition matrices of the Conway group

The double covering group of the Conway group Co_1 has nine 7-blocks of defect larger than 0. Seven of these are of defect 1 and their decomposition matrices are given in [56]. The remaining two blocks are of maximal defect 2. Each contains 29 ordinary and 21 irreducible Brauer characters.

6.1 The faithful block of maximal defect

We start with the block of maximal defect containing faithful characters, shortly called the faithful block in the following. The proof for the faithful block is easier than that for the principal block, and the result is used in the proof for the principal block.

We start with the set of projective characters displayed in Table 6.1. The origin of the projective characters is documented in table 6.4. The symbol θ_i denotes the ith character of Co_2 (in ATLAS ordering), and a bar denotes the character times the sign character of $2 \times Co_2$.

The two characters of degree 9 152 000 are complex conjugates of each other. All other characters in the block are real. Since the number of irreducible Brauer characters in the block is 21, there are exactly 19 real valued projective indecomposable characters. Each of Λ_1–Λ_{20}, except Λ_{12}, contains one of these. Therefore, Λ_{12} contains a pair of complex conjugate projective indecomposable characters. In particular, the decomposition matrix has wedge shape. From this it immediately follows, that Λ_2–Λ_4, Λ_7, Λ_9, Λ_{11} and Λ_{14}–Λ_{20} are projective indecomposable characters.

We now consider the projectives given in Table 6.2. Table 6.4 gives their origin and Table 6.3 their decompositions in terms of the projectives of Table 6.1, the relations.

Using these relations, one can reduce the first projectives to obtain a new set of projectives, given in Table 6.5. For example, Λ_{30} shows, that the PIM Λ_{18} is contained in Λ_1, and Λ_{31} shows, that the PIM Λ_3 is contained in Λ_1. This gives Φ_1 of Table 6.5. All the other new projectives are obtained by similar arguments. Since the two characters of degree 9 152 000 are complex conjugates of each other, Φ_{12} contains a pair of complex conjugate PIMs. This immediately determines the decomposition matrix.

6.2 The principal block

We start with the set of projectives displayed in Table 6.6. The origin of these is documented in Table 6.9. It is not too difficult to check that Λ_2, Λ_4, Λ_6–Λ_9, Λ_{12}, Λ_{15} and Λ_{19} are projective indecomposable characters. (For example, this can be done by checking that no subsum of these characters is zero on all 7-singular classes.)

We have displayed further projectives in Table 6.7, their origin in Table 6.9, where the notation $\Phi_{i,j}$ stands for the i-th projective indecomposable character in Block j. The numbering of the blocks follows Appendix A.2. The expression of the new projectives in terms of the first set of projectives is given in Table 6.8. Using these relations, one can reduce the basic projectives to obtain a new set of projectives, given in Table 6.10. The arguments are exactly the same as those used for the proof of Block 7.

One immediately checks that all of these, except possibly Φ_{10} and Φ_{14} are characters of projective indecomposable modules. Furthermore, either Φ_{10} is a PIM or $\Phi_{10} - \Phi_{17}$ is one. Similarly, either Φ_{14} is a PIM or $\Phi_{14} - \Phi_{20}$ is one.

Thus we are left with four possible decomposition matrices. To decide which of these is correct, we reformulate the problem in terms of Brauer characters. All irreducible Brauer characters are known, except ϕ_{17} and ϕ_{20}. The degree of ϕ_{10} is 2 038 674, that of ϕ_{14} is 10 140 998. If Φ_{10} is a PIM, then the degree of ϕ_{17} is 62 725 301, if $\Phi_{10} - \Phi_{17}$ is a PIM we have $\phi_{17}(1) = 62\,725\,301 + 2\,038\,674 = 64\,763\,975$. Similarly, if Φ_{14} is a PIM, then the degree of ϕ_{20} is 124 375 559 if $\Phi_{14} - \Phi_{20}$ is a PIM we have $\phi_{20}(1) = 124\,375\,559 + 10\,140\,998 = 134\,516\,557$.

Suppose first, that Φ_{10} is a PIM. Consider the irreducible Brauer character of Block 7 of degree 50 207 872. Denoting the irreducible Brauer characters by their degrees, we obtain

$$24 \otimes 2\,038\,674 = 48\,928\,176, \tag{6.1}$$

$$24 \otimes 50\,207\,872 = 62\,725\,301 + 2\,038\,674 + \psi, \tag{6.2}$$

where ψ is a sum of characters not in the principal block, namely $\psi = 52\,465\,644 + 256\,168\,549 + 309\,429\,120 + 522\,161\,640$. Let F denote an algebraically closed field of characteristic 7. For two FG-modules M and N, the dimension of the space of FG-homomorphisms from M to N is denoted by $[M, N]$. With this notation we have,

$$[24 \otimes 2\,038\,674, 50\,207\,872] = 0,$$

by equation (6.1), and so,

$$[2\,038\,674, 24 \otimes 50\,207\,872] = 0. \tag{6.3}$$

Since all modules considered are self-dual, equation (6.2) implies that the irreducible 2 038 674 is in the socle of $24 \otimes 50\,207\,872$, contradicting equation (6.3). We have proved that Φ_{10} is decomposable.

Now suppose that Φ_{14} is a PIM. This time we consider the irreducible Brauer character of degree 10 039 568 of Block 7. We obtain:

$$24 \otimes 10\,140\,998 = 243\,383\,952,$$

$$24 \otimes 10\,039\,568 = 124\,375\,559 + 10\,140\,998 + \psi, \tag{6.4}$$

where ψ is a sum of characters not in the principal block, namely $\psi = 77\,700\,854 + 25\,892\,020 + 2\,840\,201$. However, as above,

$$[10\,140\,998, 24 \otimes 10\,039\,568] = 0,$$

contradicting equation (6.4). We have shown that Φ_{14} decomposes, and thus completed the proof of the decomposition matrix.

6.3 Tables

Here we collect the tables of projective characters which are needed in the previous sections.

Table 6.1: Some projectives in the faithful block

Λ :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
24	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2024	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4576	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
40480	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
95680	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
299000	1	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
315744	1	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
789360	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.
1937520	1	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
5051904	.	.	.	1	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.
7104240	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
9152000	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
9152000	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
13156000	.	1	.	1	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.
17050176	.	1	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.
19734000	1	.	.	.	.	.	.	.	1	2	.	.	.	1	.	.	.	.	.	.
34155000	1	.	1	.	.	1	1	.	.	2	.	.	.	.	1	.	.	.	.	.
49335000	.	.	.	.	2	1	1	.	.	.	.	.	.	.	.	1	.	.	.	.
50519040	.	.	.	.	.	.	1	.	.	.	.	1	1	.	.	.	1	.	.	.
67358720	.	.	.	.	.	1	.	1	.	3	.	2	.	1	1	.	.	.	.	.
210496000	1	.	.	.	1	1	.	.	.	.	1	.	1	.	.	.	.	1	.	.
215547904	.	.	.	.	.	1	.	3	.	1	.	6	2	.	.	.	.	.	1	.
313524224	1	.	.	.	.	.	.	2	.	4	.	4	.	1	1	.	.	.	.	1
394680000	.	.	.	.	1	1	1	1	.	2	.	5	1	.	1	1	1	.	.	1
485760000	2	.	.	.	2	2	.	1	.	.	.	2	.	.	.	1	.	1	.	1
517899096	.	.	.	.	.	1	.	5	.	1	.	11	2	.	.	.	1	.	1	1
655360000	1	.	.	.	.	2	.	4	.	2	.	8	2	.	.	.	.	1	1	1

Table 6.2: More projectives in the faithful block

Λ :	21	22	23	24	25	26	27	28	29	30	31	32
24	.	.	.	.	.	.	.	.	.	1	1	.
2024	.	.	.	.	.	.	.	.	.	1	.	.
4576	.	.	.	.	.	.	.	.	.	2	.	.
40480	.	.	.	.	.	1	.	.	.	1	.	.
95680	.	.	.	.	1	.	.	.	.	1	1	.
299000	2	.	.	.	.	3	.	.	.	4	.	.
315744	.	.	.	.	1	.	.	.	.	2	.	.
789360	2	.	.	.	.	4	.	.	.	2	.	1
1937520	3	1	.	.	.	7	.	.	1	5	1	.
5051904	3	2	.	1	.	9	1	1	1	4	.	1
7104240	.	.	2	.	2	.	.	.	.	.	.	.
9152000	.	.	.	.	.	.	1	1	1	.	.	.
9152000	.	.	.	.	.	.	1	1	1	.	.	.
13156000	.	1	1	2	.	2	2	2	.	1	.	1
17050176	.	.	3	1	1	.	1	1	.	1	.	.
19734000	4	5	.	3	.	14	2	1	2	6	1	2
34155000	2	4	.	4	1	7	.	1	1	6	1	1
49335000	.	.	.	.	2	.	.	.	.	.	2	.
50519040	.	.	.	.	1	.	2	1	.	.	.	.
67358720	3	7	.	6	.	14	3	3	4	5	1	3
210496000	.	.	6	1	2	.	4	3	.	.	4	.
215547904	.	.	2	1	.	.	11	6	1	.	1	1
313524224	1	10	1	10	.	13	8	7	5	4	6	4
394680000	.	6	1	7	1	5	8	7	4	2	7	1
485760000	.	2	4	3	1	1	7	5	1	1	11	.
517899096	.	2	2	3	.	2	18	11	4	.	5	2
655360000	.	3	6	5	.	2	19	12	2	.	9	2

Table 6.3: Relations in the faithful block

$\Lambda:$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Λ_{21}	.	.	.	.	.	2	.	.	3	.	.	.	.	1	.	−2	.	−2	−2	.
Λ_{22}	.	.	.	.	.	.	.	.	1	1	.	.	.	2	2	.	.	.	−1	2
Λ_{23}	.	.	.	.	.	.	.	.	.	.	2	.	1	.	.	.	−1	3	.	1
Λ_{24}	.	.	.	.	.	.	.	.	.	1	.	.	1	1	2	.	−1	.	−2	3
Λ_{25}	.	.	.	.	1	.	1	.	.	.	1	.	.	.	.	−1	.	.	.	.
Λ_{26}	.	.	.	1	.	3	.	1	7	1	.	.	.	5	2	−3	.	−3	−7	.
Λ_{27}	.	.	.	.	.	.	.	.	.	1	.	1	1	.	−2	.	.	3	2	2
Λ_{28}	.	.	.	.	.	.	.	.	.	1	.	1	1	−1	−1	.	−1	2	−3	1
Λ_{29}	.	.	.	.	.	.	.	.	1	.	.	1	.	1	1	.	−1	.	−5	−1
Λ_{30}	1	1	1	.	.	2	.	.	4	.	.	.	.	1	2	−2	.	−3	−2	.
Λ_{31}	1	.	−1	.	.	.	.	.	.	.	.	.	.	.	1	2	.	3	1	4
Λ_{32}	.	.	.	.	.	.	.	1	.	1	.	.	.	.	−1	.	.	.	−3	−1

Table 6.4: Origin of projective characters of the faithful block

Char.	Origin	Char.	Origin
Λ_1 :	$\mathrm{Ind}_{2\mathrm{xCo}2}(\bar{\theta}_2+\bar{\theta}_{21})$	Λ_{17} :	$\chi_2\otimes\chi_{118}$
Λ_2 :	$\mathrm{Ind}_{2\mathrm{xCo}2}(\bar{\theta}_5)$	Λ_{18} :	$\chi_2\otimes\chi_{128}$
Λ_3 :	$\chi_3\otimes\chi_{111}$	Λ_{19} :	$\chi_2\otimes\chi_{125}$
Λ_4 :	$\mathrm{Ind}_{2\mathrm{xCo}2}(\bar{\theta}_9)$	Λ_{20} :	$\chi_{102}\otimes\chi_{75}$
Λ_5 :	$\mathrm{Ind}_{2\mathrm{xCo}2}(\bar{\theta}_{18})$	Λ_{21} :	$\chi_2\otimes\chi_{119}$
Λ_6 :	$\mathrm{Ind}_{2\mathrm{xCo}2}(\bar{\theta}_{15}+\bar{\theta}_{24})$	Λ_{22} :	$\chi_2\otimes\chi_{147}$
Λ_7 :	$\chi_{104}\otimes\chi_{16}$	Λ_{23} :	$\chi_2\otimes\chi_{148}$
Λ_8 :	$\mathrm{Ind}_{2\mathrm{xCo}2}(\bar{\theta}_{22}+\bar{\theta}_{23}+\bar{\theta}_{24})$	Λ_{24} :	$\chi_2\otimes\chi_{149}$
Λ_9 :	$\chi_{102}\otimes\chi_{36}$	Λ_{25} :	$\chi_3\otimes\chi_{131}$
Λ_{10} :	$\chi_2\otimes\chi_{141}$	Λ_{26} :	$\chi_4\otimes\chi_{132}$
Λ_{11} :	$\chi_{102}\otimes\chi_{39}$	Λ_{27} :	$\mathrm{Ind}_{2\mathrm{xCo}2}(\bar{\theta}_{50})$
Λ_{12} :	$\chi_5\otimes\chi_{118}$	Λ_{28} :	$\chi_3\otimes\chi_{156}$
Λ_{13} :	$\chi_{104}\otimes\chi_{33}$	Λ_{29} :	$\chi_4\otimes\chi_{127}$
Λ_{14} :	$\chi_{102}\otimes\chi_{49}$	Λ_{30} :	$\chi_7\otimes\chi_{111}$
Λ_{15} :	$\chi_2\otimes\chi_{127}$	Λ_{31} :	$\mathrm{Ind}_{2\mathrm{xCo}2}(\bar{\theta}_1+\bar{\theta}_{44})$
Λ_{16} :	$\chi_3\otimes\chi_{118}$	Λ_{32} :	$\chi_3\otimes\chi_{141}$

Table 6.5: Refined projectives in the faithful block

Φ :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
24	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2024	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4576	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
40480	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
95680	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
299000	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
315744	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
789360	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.
1937520	1	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
5051904	.	.	.	1	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.
7104240	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
9152000	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
9152000	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
13156000	.	1	.	1	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.
17050176	.	1	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.
19734000	1	.	.	.	.	.	.	.	1	1	.	.	.	1	.	.	.	.	.	.
34155000	.	.	1	.	.	1	1	.	.	.	.	.	.	.	1	.	.	.	.	.
49335000	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	1	.	.	.	.
50519040	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.
67358720	.	.	.	.	.	1	.	1	.	.	.	2	.	1	1	.	.	.	.	.
210496000	.	.	.	.	1	.	.	.	.	.	1	.	1	.	.	.	.	1	.	.
215547904	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	1	.
313524224	1	.	.	.	.	.	.	1	.	1	.	3	.	1	1	.	.	.	.	1
394680000	.	.	.	.	.	.	1	.	.	.	.	3	.	.	1	1	1	.	.	1
485760000	1	.	.	.	1	.	.	.	.	.	.	1	.	.	.	1	.	1	.	1
517899096	.	.	.	.	.	.	.	1	.	.	.	4	.	.	.	.	1	.	1	1
655360000	.	.	.	.	.	.	.	.	.	1	.	2	1	.	.	.	.	1	1	1

Table 6.6: Basic set of projectives of the principal block

Λ :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
276	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
299	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
17250	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
80730	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
94875	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
822250	1	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
871884	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
1821600	1	1	.	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
2055625	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
9221850	1	.	.	.	.	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.
16347825	.	.	.	1	2	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.
21528000	1	.	.	.	.	1	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.
21579129	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.
24667500	1	.	.	.	1	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.
31574400	2	.	.	.	.	.	.	1	.	.	.	.	1	1	.	.	.	.	.	.	.
57544344	2	.	1	1	.	.	1	.	.	.	1	.	.	1	.	1	.	.	.	.	.
66602250	.	.	.	2	1	.	.	.	1	1	.	.	.	1	.	.	2	1	1	.	.
85250880	2	.	1	2	.	.	.	.	.	1	.	.	.	2	.	1	1	.	.	.	.
150732800	.	.	.	1	.	.	.	.	.	1	.	.	1	1	.	1	1	1	.	1	1
163478250	3	1	.	1	.	1	.	.	.	.	1	.	1	1	.	1	1	.	.	1	1
191102976	1	.	.	1	.	.	.	.	.	.	.	1	1	2	1	1	1	.	.	1	1
207491625	1	.	.	1	.	.	.	.	.	1	.	.	1	3	.	.	3	1	1	1	.
215547904	1	1	.	2	1	.	.	.	1	1	.	1	.	2	.	1	3	1	1	1	1
219648000	.	.	.	.	.	.	.	.	.	.	.	.	2	1	1	.	1	1	.	2	1
299710125	2	.	.	1	.	.	.	.	.	.	1	.	2	3	.	1	2	1	.	2	1
326956500	1	.	.	1	.	.	.	.	.	1	1	.	1	3	1	1	3	1	1	2	1

Table 6.7: More projectives in the principal block

Λ :	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
1	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.
276	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
299	.	2	.	.	1	.	.	.	.	.	.	.	.	.	.	.
17250	1	2	.	.	1	.	1	.	.	.	.	.	.	.	.	.
80730	2	1	.	.	1	.	2	.	.	.	.	.	.	.	.	.
94875	.	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.
822250	.	8	.	.	1	.	.	.	.	.	.	.	.	.	.	1
871884	4	6	.	.	1	.	.	.	.	.	.	.	.	.	.	.
1821600	.	1	.	.	.	.	1	.	.	.	.	.	1	.	.	.
2055625	3	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
9221850	.	11	.	.	.	.	.	.	.	.	1	2	.	1	2	1
16347825	4	1	1	1	1	.	3	.	.	1	.	.	1	.	.	.
21528000	5	13	.	.	.	.	.	.	.	.	1	2	.	1	2	1
21579129	1	.	1	1	.	1	.	1	1	.	.	.	.	1	.	.
24667500	6	1	2	2	2	1	2	1	.	1	.	.	.	.	.	.
31574400	7	12	1	1	1	1	.	1	.	.	1	2	.	1	2	1
57544344	5	19	.	1	2	1	.	1	.	.	3	3	.	2	2	2
66602250	10	1	2	3	2	2	2	.	.	1	.	.	1	.	.	.
85250880	14	10	1	3	4	1	1	3	.	1	3	1	.	2	1	1
150732800	13	12	3	2	1	2	.	1	.	.	2	1	.	2	.	1
163478250	5	18	3	1	1	1	.	2	.	.	4	4	1	3	4	1
191102976	9	13	5	3	2	2	.	3	1	1	3	2	1	3	2	1
207491625	20	1	7	5	4	3	1	4	2	2	1	.	1	2	1	.
215547904	15	8	6	5	4	3	1	2	.	2	2	.	2	1	.	.
219648000	10	5	7	1	.	2	.	2	3	.	.	.	1	3	.	.
299710125	21	16	8	4	3	3	.	5	2	1	4	3	1	5	3	1
326956500	16	11	9	4	3	4	.	4	3	1	3	2	2	4	2	.

Table 6.8: Relations in the principal block

Λ :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Λ_{22}	1	−1	−1	1	1	.	.	3	−2	2	−1	3	1	1	.	3	4	7	−9	2	−8
Λ_{23}	1	−1	1	.	.	2	6	5	1	.	2	.	5	.	.	8	−1	.	2	−4	4
Λ_{24}	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	−1	.	.	1	3	.
Λ_{25}	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1	1	−1	−1	.
Λ_{26}	1	−1	.	.	.	.	.	.	.	.	−1	1	−1	.	.	1	1	1	−1	1	−2
Λ_{27}	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	−1	1	2	.	1
Λ_{28}	.	.	.	1	2	.	.	.	−2	.	.	.	.	.	.	−1	.	.	.	.	.
Λ_{29}	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	−1	−2	1	−1
Λ_{30}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	2	−2
Λ_{31}	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	−1	.	−1
Λ_{32}	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	2	1	−1	−1	−1	.
Λ_{33}	.	.	.	.	.	.	.	.	.	.	2	.	2	.	.	1	.	−1	1	−2	1
Λ_{34}	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	−1	1	1	.
Λ_{35}	.	.	.	.	.	.	.	.	.	.	1	.	1	.	1	1	1	.	−2	.	−1
Λ_{36}	.	.	.	.	.	.	.	.	.	.	2	.	2	.	.	.	1	−2	.	−2	1
Λ_{37}	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	1	.	.	.	−1	.

Table 6.9: Origin of projective characters of the principal block

Char.	Origin
Λ_1 :	$\chi_{102} \otimes \Phi_{1,7}$
Λ_2 :	$\mathrm{Ind}_{2\mathrm{xCo2}}(\theta_3 + \theta_{16})$
Λ_3 :	$\chi_{102} \otimes \Phi_{1,8}$
Λ_4 :	$\chi_{102} \otimes \Phi_{7,7}$
Λ_5 :	$\chi_2 \otimes \chi_{39}$
Λ_6 :	$\chi_{102} \otimes \Phi_{4,7}$
Λ_7 :	$\chi_{102} \otimes \chi_{119}$
Λ_8 :	$\chi_2 \otimes \chi_{22}$
Λ_9 :	$\chi_2 \otimes \chi_{32}$
Λ_{10} :	$\chi_6 \otimes \chi_{17}$
Λ_{11} :	$\chi_{102} \otimes \chi_{141}$
Λ_{12} :	$\chi_4 \otimes \chi_{33}$
Λ_{13} :	$\chi_2 \otimes \chi_{44}$
Λ_{14} :	$\mathrm{Ind}_{2\mathrm{xCo2}}(\theta_{43})$
Λ_{15} :	$\chi_4 \otimes \chi_{26}$
Λ_{16} :	$\chi_{102} \otimes \chi_{149}$
Λ_{17} :	$\chi_3 \otimes \chi_{73}$
Λ_{18} :	$\chi_4 \otimes \chi_{32}$
Λ_{19} :	$(\chi_2 \otimes \chi_{70})/2$

Char.	Origin
Λ_{20} :	$\chi_{102} \otimes \chi_{161}$
Λ_{21} :	$\chi_2 \otimes \chi_{46}$
Λ_{22} :	χ_{16}^{2+}
Λ_{23} :	χ_{112}^{2+}
Λ_{24} :	$\chi_7 \otimes \chi_{33}$
Λ_{25} :	$\mathrm{Ind}_{2\mathrm{xCo2}}(\theta_{45})$
Λ_{26} :	$\mathrm{Ind}_{2\mathrm{xCo2}}(\theta_1 + \theta_{44})$
Λ_{27} :	$\mathrm{Ind}_{2\mathrm{xCo2}}(\theta_{27} + \theta_{34})$
Λ_{28} :	$\mathrm{Ind}_{2\mathrm{xCo2}}(\theta_6 + \theta_{17})$
Λ_{29} :	$\chi_{102} \otimes \Phi_{21,7}$
Λ_{30} :	$\chi_{102} \otimes \Phi_{20,7}$
Λ_{31} :	$\chi_{102} \otimes \Phi_{19,7}$
Λ_{32} :	$\chi_{102} \otimes \Phi_{16,7}$
Λ_{33} :	$\chi_{102} \otimes \Phi_{15,7}$
Λ_{34} :	$\chi_{102} \otimes \Phi_{14,7}$
Λ_{35} :	$\chi_{102} \otimes \Phi_{13,7}$
Λ_{36} :	$\chi_{102} \otimes \Phi_{10,7}$
Λ_{37} :	$\chi_{102} \otimes \Phi_{3,8}$

Table 6.10: Refined basic set of projectives of the principal block

Φ :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
276	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
299	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
17250	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
80730	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
94875	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
822250	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
871884	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
1821600	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
2055625	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
9221850	.	.	.	.	.	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.
16347825	.	.	.	.	1	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.
21528000	.	.	.	.	.	1	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.
21579129	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.
24667500	1	.	.	.	1	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.
31574400	1	.	.	.	.	.	.	1	.	.	.	.	1	1	.	.	.	.	.	.	.
57544344	.	.	1	.	.	.	1	.	.	.	1	.	.	.	.	1	.	.	.	.	.
66602250	.	.	.	1	.	.	.	.	1	1	.	.	.	.	.	.	1	.	.	.	.
85250880	.	.	1	1	.	.	.	.	.	1	.	.	.	.	.	1	.	1	.	.	.
150732800	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	1	.	.
163478250	.	1	.	.	.	1	.	.	.	.	1	.	1	1	.	.	.	.	.	1	.
191102976	.	.	.	.	.	.	.	.	.	.	.	1	1	2	1	.	.	.	.	1	.
207491625	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	1	1	.	.	1
215547904	.	1	.	.	.	.	.	.	1	1	.	1	.	1	.	.	1	.	.	1	.
219648000	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	1
299710125	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	1	1	.	1
326956500	.	.	.	.	.	.	.	.	.	1	1	.	.	1	1	.	1	.	.	1	1

Appendix A

A.1 The 5-decomposition numbers of Co_2

1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
23	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
253	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
1771	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
2024	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
2277	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.
7084	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.
10395	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
10395	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
31878	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.
37422	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.
129536	.	.	1	1	.	.	1	.	.	.	1	.	.	.	.	.
184437	1	.	.	.	1	.	1	.	.	.	.	1	.	.	.	.
212520	.	.	.	.	.	2	.	.	1	.	.	.	1	.	.	.
226688	1	.	.	.	1	.	.	1	.	1	.	1	.	.	.	.
239085	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
239085	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
245916	.	.	.	.	1	.	.	.	1	1	.	1	.	.	.	.
312984	.	.	.	1	.	.	2	.	.	.	1	.	1	.	.	.
368874	.	1	1	.	.	.	.	1	.	.	1	.	.	.	1	.
430353	.	.	.	.	.	1	2	.	.	.	.	.	1	1	.	.
637560	1	.	.	.	1	.	1	1	1	.	.	2	.	.	1	.
1291059	.	.	.	.	.	1	.	.	1	1	.	1	.	.	.	1
1835008	.	.	.	.	.	.	1	1	.	.	1	1	.	1	1	1
1943040	1	.	1	.	.	.	1	2	.	1	2	2	.	.	1	1
2040192	.	.	.	.	.	1	2	.	1	.	1	1	1	1	1	1
2072576	.	1	.	.	.	1	2	.	.	.	1	.	1	2	1	1

A.2 The 7-decomposition numbers of $2Co_1$

In this section we give the 7-modular decomposition matrices for the two blocks of maximal defect of $2Co_1$. The decomposition numbers are followed by tables giving the degrees of the irreducible Brauer characters as well as the prime factorizations of these degrees. The decomposition matrices for the blocks of non-maximal defect can be found in [56].

A.2.1 Decomposition matrix of Block 1

Φ:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
276	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
299	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
17250	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
80730	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
94875	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
822250	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
871884	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
1821600	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
2055625	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
9221850	.	.	.	.	.	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.
16347825	.	.	.	.	1	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.
21528000	.	.	.	.	.	1	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.
21579129	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.
24667500	1	.	.	.	1	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.
31574400	1	.	.	.	.	.	.	1	.	.	.	.	1	1	.	.	.	.	.	.	.
57544344	.	.	1	.	.	.	1	.	.	.	1	.	.	.	.	1	.	.	.	.	.
66602250	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	1	.	.	.	.
85250880	.	.	1	1	.	.	.	.	.	1	.	.	.	.	.	1	.	1	.	.	.
150732800	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	1	.	.
163478250	.	1	.	.	.	1	.	.	.	.	1	.	1	.	.	.	.	.	.	1	.
191102976	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	.	.	.	.	1	.
207491625	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	1
215547904	.	1	.	.	.	.	.	.	1	.	.	1	.	.	.	.	1	.	.	1	.
219648000	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	1
299710125	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	1	1	.	1
326956500	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	1	.	.	1	1

A.2.2 Decomposition matrix of Block 7

Φ :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
24	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2024	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4576	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
40480	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
95680	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
299000	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
315744	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
789360	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
1937520	1	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
5051904	.	.	.	1	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.
7104240	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
9152000	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
9152000	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
13156000	.	1	.	1	.	.	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.
17050176	.	1	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.
19734000	1	.	.	.	.	.	.	.	1	1	.	.	.	.	1	.	.	.	.	.	.
34155000	.	.	1	.	.	1	1	.	.	.	.	.	.	.	.	1	.	.	.	.	.
49335000	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.
50519040	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	.	.	.
67358720	.	.	.	.	.	1	.	1	.	.	.	1	1	.	1	1	.	.	.	.	.
210496000	.	.	.	.	1	.	.	.	.	.	1	.	.	1	.	.	.	.	1	.	.
215547904	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.
313524224	1	.	.	.	.	.	.	1	.	1	.	1	1	.	1	1	.	.	.	.	1
394680000	.	.	.	.	.	.	1	.	.	.	.	1	1	.	.	1	1	1	.	.	1
485760000	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	1
517899096	.	.	.	.	.	.	.	1	.	.	.	1	1	.	.	.	.	1	.	1	1
655360000	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.	1	1	1

A.2.3 Degrees of irreducible Brauer characters

The table on the left hand side lists the degrees of the irreducible Brauer characters of the principal block, the other table those of Block 7.

Char.	Degree	Factors
ϕ_1	1	1
ϕ_2	276	2^2 3 23
ϕ_3	299	13 23
ϕ_4	16951	11 23 67
ϕ_5	80729	11 41 179
ϕ_6	94599	3^2 23 457
ϕ_7	821951	13 23 2749
ϕ_8	871883	871883
ϕ_9	1821324	2^2 3 23 6599
ϕ_{10}	2038674	2 3 11 17 23 79
ϕ_{11}	8305300	2^2 5^2 23^2 157
ϕ_{12}	14445772	2^2 11 569 577
ϕ_{13}	20561518	2 607 16937
ϕ_{14}	10140998	2 7 227 3191
ϕ_{15}	11438131	11438131
ϕ_{16}	48416794	2 23 37 28447
ϕ_{17}	64763975	5^2 23 163 691
ϕ_{18}	34778162	2 23 83 9109
ϕ_{19}	100277332	2^2 23 733 1487
ϕ_{20}	134516557	7 19216651
ϕ_{21}	107932537	23 43 109133

Degree	Factors
24	2^3 3
2024	2^3 11 23
4576	2^5 11 13
38456	2^3 11 19 23
95656	2^3 11 1087
294424	2^3 13 19 149
311168	2^7 11 13 17
494936	2^3 13 4759
1937496	2^3 3 11 41 179
3075952	2^4 11 17477
7008584	2^3 11 73 1091
9152000	2^9 5^3 11 13
9152000	2^9 5^3 11 13
10039568	2^4 7 11 29 281
14720528	2^4 283 3251
33544832	2^7 262069
48928176	2^4 3^2 11 17 23 79
50207872	2^7 11 13^2 211
193352192	2^9 11^2 3121
205508336	2^4 11^2 101 1051
243383952	2^4 3 7 227 3191

Appendix B

B.1 The 5-modular character table of Co_2

;	@	@	@	@	@	@	@	@	@
	42	743	41	1	46	15	3	12	7
	305421312000	178240	287680	474560	6560	5520	096576	2880	3728
p power		A	A	A	A	A	A	B	B
p’ part		A	A	A	A	A	A	A	A
ind	1A	2A	2B	2C	3A	3B	4A	4B	4C
+	1	1	1	1	1	1	1	1	1
+	23	-9	7	-1	-4	5	7	-5	3
+	253	29	13	-11	10	10	29	9	1
+	275	51	35	11	5	14	19	15	7
+	1771	-21	-21	11	-11	16	91	-5	-5
+	2023	231	103	39	-2	25	7	23	23
+	2254	-210	126	-10	13	31	14	-30	10
+	4025	-231	105	1	-25	29	105	-35	5
+	5313	-63	-63	33	21	3	49	1	1
o	9625	-455	105	-15	40	-5	-7	5	29
o	9625	-455	105	-15	40	-5	-7	5	29
+	10395	315	-21	-45	27	0	-21	-9	15
+	12650	554	330	26	-40	59	-6	50	2
+	23000	600	280	120	50	5	184	40	8
+	29624	-168	392	-16	32	14	-56	4	-36
+	31625	265	-55	-55	35	35	377	25	-7
+	31625	1385	505	145	35	35	41	45	53
+	37169	1105	449	-55	-10	71	97	77	5
+	44275	-1869	595	-29	-5	94	35	-85	59
+	63250	-110	210	-110	-65	-20	322	-30	2

o	91125	405	-315	45	0	0	-27	-15	9
o	91125	405	-315	45	0	0	-27	-15	9
+	113575	903	-105	143	40	-23	119	-5	3
+	122199	567	-441	-33	-84	15	343	-5	3
+	173075	-973	35	43	5	-58	483	15	7
+	177100	2828	364	188	-20	-29	-84	-28	20
+	178388	-1932	420	76	53	44	-28	-48	-24
+	221375	4095	735	15	-160	-25	-49	-25	87
+	236004	-92	-188	-108	51	-30	20	-20	-4
+	239085	-2835	-147	45	-108	0	-147	45	45
+	253000	2120	-440	200	-125	10	104	-40	24
+	284625	-3855	1505	-55	-90	45	273	-115	5
+	354200	-4872	840	-32	-40	-58	-56	60	36
+	385825	3457	1105	233	-5	-23	-15	5	-19
+	442750	-770	1470	-130	-185	40	-210	30	-66
+	462000	5040	560	-400	30	120	112	80	16
+	664125	-1155	-35	-275	195	-30	637	85	5
+	664125	-1155	-35	365	195	-30	-35	5	-11
+	664125	2205	1645	-315	195	-30	77	-15	-39
+	853875	7155	435	-45	135	0	-237	-45	51
+	1044912	-720	496	-80	-15	-87	-160	16	-80
+	1288000	-2240	-2240	320	100	100	448	0	0
+	1771000	-12040	1400	-200	205	-20	-168	40	40
+	1771000	1400	-840	-40	-200	115	-168	40	-56
+	1992375	3255	-1225	215	180	45	-441	-25	-25
+	2004750	8910	-1170	-450	0	0	-162	-90	54
+	2095875	-3645	-2205	-45	0	0	-189	75	27

@	@	@	@	@	@	@	@	@	@	@	@	@
4												
9152	6144	6144	1280	5760	5184	4320	3456	576	288	56	768	768
A	B	B	C	AB	AA	BA	BA	BB	BC	A	A	C
A	A	A	A	AB	AA	BA	BA	BB	BC	A	A	A
4D	4E	4F	4G	6A	6B	6C	6D	6E	6F	7A	8A	8B
1	1	1	1	1	1	1	1	1	1	1	1	1
-1	3	-1	-1	4	0	3	-3	1	-1	2	-1	-3
5	1	-3	1	10	2	2	2	-2	-2	1	-3	3
-5	7	3	-1	5	-3	6	6	2	2	2	3	5
-5	-5	3	-1	21	-3	-6	0	0	2	0	3	-1
7	7	7	-1	-2	6	3	9	1	3	0	-1	3
6	10	-2	2	-3	-3	3	-9	3	-1	0	-2	-4
1	5	1	1	15	3	9	-3	-3	1	0	1	-5
17	1	-7	-3	-3	9	-3	3	3	-3	0	1	1
1	-3	-7	5	0	4	-5	-5	3	3	0	1	3
1	-3	-7	5	0	4	-5	-5	3	3	0	1	3
19	-1	3	-5	3	-9	0	0	0	0	0	3	1
10	18	2	6	0	-4	-1	11	3	-1	1	2	6
24	8	8	0	10	6	15	-3	1	3	-2	8	0
-16	12	0	-8	8	12	-18	6	2	2	0	0	2
-7	-7	1	5	35	-5	-5	-5	-1	-1	-1	-7	1
1	5	25	5	-5	-1	5	11	7	1	-1	1	-1
-23	5	-7	-7	-10	-2	7	7	-1	-1	-1	1	3
-13	11	-5	-1	-5	3	6	-18	-2	-2	0	3	-5
2	2	10	-10	15	7	10	4	0	-2	-2	-6	2

-51	9	-3	5	0	0	0	0	0	0	-1	-3	3
-51	9	-3	5	0	0	0	0	0	0	-1	-3	3
-17	3	-1	7	0	12	3	-15	-3	-1	0	-1	-7
-17	3	7	3	-12	0	-9	-9	3	3	0	-1	-3
-21	7	-21	3	5	-19	2	14	2	-2	0	3	5
-4	-12	-4	12	4	-16	-1	-13	-5	-1	0	-4	-8
12	-24	12	4	-3	-15	12	12	-12	4	0	4	2
-1	-9	7	-5	0	0	-15	15	3	-3	0	-1	-9
36	12	12	-8	-5	7	4	10	-2	0	-1	-4	8
-19	-3	-3	5	12	0	0	0	0	0	0	-3	-3
40	8	-24	0	-5	23	-10	2	-2	2	-1	0	4
-7	5	1	5	-10	6	15	-3	5	-1	-2	-7	3
32	-12	-16	-8	0	-12	-12	6	6	4	0	0	10
9	-19	25	-7	-5	19	7	-23	1	-1	-1	1	-5
46	14	6	10	15	-5	-20	-8	0	-4	0	6	2
48	-16	-16	0	-10	18	0	0	-4	-4	0	0	0
13	5	-3	5	-5	15	0	-6	-2	4	0	-3	-7
45	5	-3	5	-5	15	0	-6	-2	-4	0	-3	5
37	-7	-11	5	-5	-9	0	18	10	0	0	-3	-5
-77	-13	3	-5	15	27	0	0	0	0	1	3	-1
-32	0	-8	-4	1	9	9	9	1	1	1	8	4
-64	0	0	0	-20	-8	-20	4	4	-4	0	0	0
-40	-8	24	0	5	-7	-10	-4	-4	-2	0	0	4
-8	-24	8	0	0	-4	5	-13	3	5	0	-8	0
39	-9	-17	-5	20	-12	15	-3	5	-1	0	-1	3
-18	6	6	10	0	0	0	0	0	0	-1	6	6
51	27	3	-5	0	0	0	0	0	0	-2	3	-9

@	@	@	@	@	@	@	@	@	@	@	@	@
512	512	256	64	54	11	864	288	288	288	96	96	48
D	D	C	E	A	A	BA	AC	DA	EC	BD	EB	AE
A	A	A	A	A	A	AA	AC	BA	BC	AD	BB	AE
8C	8D	8E	8F	9A	11A	12A	12B	12C	12D	12E	12F	12G
1	1	1	1	1	1	1	1	1	1	1	1	1
3	-1	1	1	2	1	-2	0	1	-3	2	1	0
5	1	-1	-1	1	0	2	-2	2	4	2	0	-2
3	-1	1	1	2	0	1	1	-2	4	1	0	1
7	-1	-1	-1	-2	0	1	1	4	-2	1	-2	1
-1	-1	3	-1	-2	-1	-2	2	1	-1	-2	-1	-2
-2	2	0	0	-2	-1	5	1	-1	1	-3	-3	1
5	1	-1	-1	2	-1	-3	-1	-3	-7	1	1	-1
-3	5	1	1	0	0	-5	1	-5	1	-1	1	1
1	-3	-1	-1	1	0	2	-4	-1	-1	-2	-1	0
1	-3	-1	-1	1	0	2	-4	-1	-1	-2	-1	0
-1	3	-3	1	0	0	-3	3	0	0	1	0	-1
-2	-2	-2	2	-1	0	-6	-4	3	-1	-2	-1	0
0	0	0	0	2	-1	4	2	1	5	0	1	2
-4	0	-2	2	2	1	-2	0	-2	0	2	4	0
5	-3	1	1	-1	0	-1	-1	-1	5	-1	1	-1
1	5	3	-1	-1	0	5	-1	-1	-1	1	3	-1
-3	1	-1	-1	-1	0	-2	2	-5	-1	-2	-1	2
-1	-1	3	-1	-2	0	-1	-1	2	2	-1	2	-1
-2	6	2	2	1	0	7	-1	4	-4	-1	0	-1

1	5	-1	-1	0	1	0	0	0	0	0	0	0
1	5	-1	-1	0	1	0	0	0	0	0	0	0
-5	7	-3	-3	-2	0	2	0	5	-3	-2	1	0
-9	-5	1	1	0	0	10	0	-5	-3	-2	1	0
-9	-5	1	1	-1	1	-3	1	6	4	-3	0	1
4	-4	0	0	1	0	6	-4	3	-1	2	-1	0
8	-4	-2	-2	2	1	-1	-3	8	0	3	0	-3
3	3	-1	-1	2	0	-4	0	-1	3	-4	-1	0
0	0	0	-4	-3	-1	-7	-1	2	2	-3	-2	3
-3	-3	-3	1	0	0	6	0	0	0	2	0	0
0	0	4	0	1	0	5	3	2	0	1	-4	-1
1	-3	-1	-1	0	0	-6	2	-3	-1	2	-1	2
4	0	-2	2	2	0	-2	0	-2	0	2	0	0
5	1	-1	-1	1	0	3	-1	-3	-1	3	-1	-1
-6	2	2	-2	1	0	-3	3	0	0	1	0	-1
0	0	0	0	0	0	4	-2	4	-2	0	2	2
-3	5	1	1	0	0	-11	-1	-2	2	1	-2	-1
5	-3	-3	1	0	0	1	7	-2	-2	-3	2	-1
1	-3	-1	-1	0	0	5	3	2	0	1	0	-1
-5	-5	-1	-1	0	0	-3	3	0	0	1	0	-1
12	4	4	0	0	0	-7	1	5	1	1	1	-3
0	0	0	0	1	-1	-2	0	4	0	2	0	0
0	0	4	0	1	0	3	1	0	-2	-1	-2	1
0	0	0	0	1	0	-6	4	3	1	-2	1	0
-5	3	3	-1	0	0	0	-4	-3	-1	0	-1	0
2	2	-2	2	0	0	0	0	0	0	0	0	0
3	-5	-1	-1	0	1	0	0	0	0	0	0	0

@	@	@	@	@	@	@	@	@	@	@	@
48	56	28	28	32	32	18	23	23	24	24	28
EF	AA	AB	AB	D	C	AB	A	A	CA	BB	AA
BF	AA	AB	AB	A	A	AA	A	A	BA	AB	AA
12H	14A	14B	C**	16A	16B	18A	23A	B**	24A	24B	28A
1	1	1	1	1	1	1	1	1	1	1	1
-1	-2	0	0	1	-1	0	0	0	-1	0	0
0	1	-1	-1	1	-1	-1	0	0	0	0	1
0	2	0	0	-1	1	0	-1	-1	0	-1	-2
0	0	0	0	1	1	0	0	0	0	-1	0
1	0	-2	-2	-1	-1	0	-1	-1	-1	0	0
1	0	0	0	-2	0	0	0	0	1	-1	0
1	0	0	0	-1	1	0	0	0	1	1	0
-1	0	0	0	-1	-1	0	0	0	1	1	0
-1	0	0	0	1	-1	1	b23	**	1	0	0
-1	0	0	0	1	-1	1	**	b23	1	0	0
0	0	0	0	1	-1	0	-1	-1	0	1	0
-1	1	1	1	0	0	-1	0	0	-1	0	1
-1	-2	0	0	0	0	0	0	0	-1	0	2
0	0	0	0	2	0	0	0	0	0	2	0
1	-1	1	1	-1	-1	1	0	0	-1	1	-1
1	-1	1	1	1	-1	-1	0	0	1	-1	-1
-1	-1	1	1	-1	1	1	1	1	1	0	-1
-2	0	0	0	1	1	0	0	0	0	1	0
-2	2	0	0	0	0	1	0	0	0	-1	0

0	-1	i7	-i7	-1	1	0	-1	-1	0	0	1
0	-1	-i7	i7	-1	1	0	-1	-1	0	0	1
-1	0	0	0	1	-1	0	1	1	-1	2	0
1	0	0	0	-1	1	0	0	0	-1	0	0
0	0	0	0	1	-1	-1	0	0	0	-1	0
-1	0	0	0	0	0	-1	0	0	-1	-2	0
0	0	0	0	2	0	0	0	0	-2	-1	0
1	0	0	0	1	1	0	0	0	-1	0	0
0	-1	1	1	-2	2	1	1	1	2	-1	-1
0	0	0	0	1	1	0	0	0	0	0	0
0	-1	1	1	0	0	-1	0	0	0	1	-1
1	2	0	0	1	-1	0	0	0	-1	0	0
2	0	0	0	-2	0	0	0	0	0	-2	0
1	-1	-1	-1	-1	1	1	0	0	1	1	-1
0	0	0	0	0	0	1	0	0	0	-1	0
2	0	0	0	0	0	0	-1	-1	0	0	0
0	0	0	0	1	1	0	0	0	0	-1	0
0	0	0	0	1	1	0	0	0	0	-1	0
-2	0	0	0	-1	1	0	0	0	0	1	0
0	1	1	1	-1	-1	0	0	0	0	-1	1
1	1	-1	-1	2	-2	0	-1	-1	-1	1	1
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	-1	0	0	0	1	0
-1	0	0	0	0	0	-1	0	0	1	0	0
1	0	0	0	1	1	0	0	0	-1	0	0
0	-1	-1	-1	0	0	0	1	1	0	0	-1
0	2	0	0	-1	-1	0	0	0	0	0	0

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